18. The total revenue in Rupees received from the sale of x units of a product is given by
$R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is
(A) 116 (B) 96 (C) 90 (D) 126

6.3 Increasing and Decreasing Functions

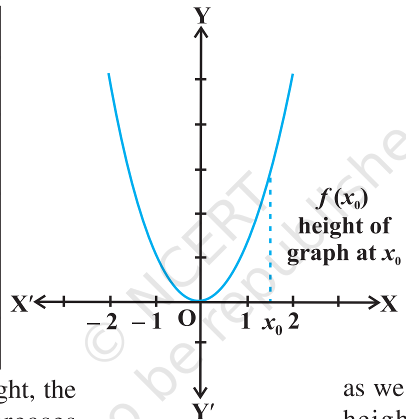
In this section, we will use differentiation to find out whether a function is increasing or decreasing or none.

Consider the function f given by $f(x) = x^2$, $x \in \mathbf{R}$. The graph of this function is a parabola as given in Fig 6.1.

Values left to origin

x	$f(x) = x^2$
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$
-1	1
$-\frac{1}{2}$	$\frac{1}{4}$
0	0

as we move from left to right, the height of the graph decreases



Values right to origin

x	$f(x) = x^2$
0	0
$\frac{1}{2}$	$\frac{1}{4}$
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

as we move from left to right, the height of the graph increases

Fig 6.1

First consider the graph (Fig 6.1) to the right of the origin. Observe that as we move from left to right along the graph, the height of the graph continuously increases. For this reason, the function is said to be increasing for the real numbers $x > 0$.

Now consider the graph to the left of the origin and observe here that as we move from left to right along the graph, the height of the graph continuously decreases. Consequently, the function is said to be decreasing for the real numbers $x < 0$.

We shall now give the following analytical definitions for a function which is increasing or decreasing on an interval.

Definition 1 Let I be an interval contained in the domain of a real valued function f. Then f is said to be

(i) increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

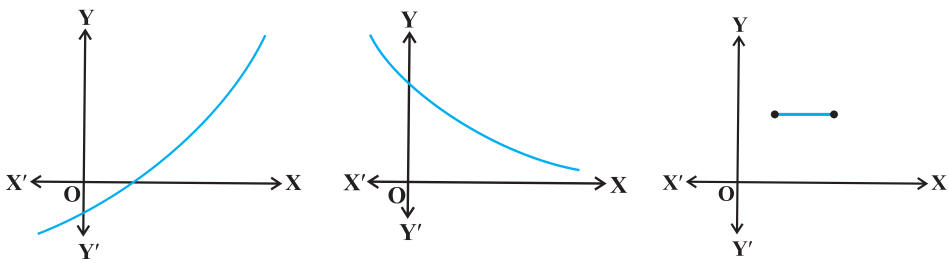
(ii) decreasing on I, if x_1, x_2 in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

(iii) constant on I, if $f(x) = c$ for all $x \in$ I, where c is a constant.

(iv) decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in$ I.

(v) strictly decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in$ I.

For graphical representation of such functions see Fig 6.2.



Strictly Increasing function
(i)

Strictly Decreasing function
(ii)

Neither Increasing nor Decreasing function
(iii)

Fig 6.2

We shall now define when a function is increasing or decreasing at a point.

Definition 2 Let x_0 be a point in the domain of definition of a real valued function f. Then f is said to be increasing, decreasing at x_0 if there exists an open interval I containing x_0 such that f is increasing, decreasing, respectively, in I.

Let us clarify this definition for the case of increasing function.

Example 7 Show that the function given by $f(x) = 7x - 3$ is increasing on **R**.

Solution Let x_1 and x_2 be any two numbers in **R**. Then

$$x_1 < x_2 \Rightarrow 7x_1 < 7x_2 \Rightarrow 7x_1 - 3 < 7x_2 - 3 \Rightarrow f(x_1) < f(x_2)$$

Thus, by Definition 1, it follows that f is strictly increasing on **R**.

We shall now give the first derivative test for increasing and decreasing functions. The proof of this test requires the Mean Value Theorem studied in Chapter 5.

Theorem 1 Let f be continuous on $[a, b]$ and differentiable on the open interval (a,b). Then

(a) f is increasing in $[a,b]$ if $f'(x) > 0$ for each $x \in (a, b)$

(b) f is decreasing in $[a,b]$ if $f'(x) < 0$ for each $x \in (a, b)$

(c) f is a constant function in $[a,b]$ if $f'(x) = 0$ for each $x \in (a, b)$

Proof (a) Let $x_1, x_2 \in [a, b]$ be such that $x_1 < x_2$.

Then, by Mean Value Theorem (Theorem 8 in Chapter 5), there exists a point c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)\ (x_2 - x_1)$$

i.e. $$f(x_2) - f(x_1) > 0 \qquad \text{(as } f'(c) > 0 \text{ (given))}$$

i.e. $$f(x_2) > f(x_1)$$

Thus, we have

$$x_1 < x_2 \quad f(x_1) \quad f(x_2), \text{ for all } x_1, x_2 \quad [a,b]$$

Hence, f is an increasing function in $[a,b]$.

The proofs of part (b) and (c) are similar. It is left as an exercise to the reader.

Remarks

There is a more generalised theorem, which states that if $f¢(x) > 0$ for x in an interval excluding the end points and f is continuous in the interval, then f is increasing. Similarly, if $f¢(x) < 0$ for x in an interval excluding the end points and f is continuous in the interval, then f is decreasing.

Example 8 Show that the function f given by

$$f(x) = x^3 - 3x^2 + 4x,\ x \in \mathbf{R}$$

is increasing on **R**.

Solution Note that

$$f'(x) = 3x^2 - 6x + 4$$
$$= 3(x^2 - 2x + 1) + 1$$
$$= 3(x - 1)^2 + 1 > 0, \text{ in every interval of } \mathbf{R}$$

Therefore, the function f is increasing on **R**.

Example 9 Prove that the function given by $f(x) = \cos x$ is

(a) decreasing in $(0, \pi)$

(b) increasing in $(\pi, 2\pi)$, and

(c) neither increasing nor decreasing in $(0, 2\pi)$.

Solution Note that $f'(x) = -\sin x$

(a) Since for each $x \in (0, \pi)$, $\sin x > 0$, we have $f'(x) < 0$ and so f is decreasing in $(0, \pi)$.

(b) Since for each $x \in (\pi, 2\pi)$, $\sin x < 0$, we have $f'(x) > 0$ and so f is increasing in $(\pi, 2\pi)$.

(c) Clearly by (a) and (b) above, f is neither increasing nor decreasing in $(0, 2\pi)$.

Example 10 Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is

(a) increasing (b) decreasing

Solution We have

Fig 6.3

$$f(x) = x^2 - 4x + 6$$

or $$f'(x) = 2x - 4$$

Therefore, $f'(x) = 0$ gives $x = 2$. Now the point $x = 2$ divides the real line into two disjoint intervals namely, $(-\infty, 2)$ and $(2, \infty)$ (Fig 6.3). In the interval $(-\infty, 2)$, $f'(x) = 2x - 4 < 0$.

Therefore, f is decreasing in this interval. Also, in the interval $(2, \infty)$, $f'(x) > 0$ and so the function f is increasing in this interval.

Example 11 Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is (a) increasing (b) decreasing.

Solution We have

$$f(x) = 4x^3 - 6x^2 - 72x + 30$$

or
$$\begin{aligned} f'(x) &= 12x^2 - 12x - 72 \\ &= 12(x^2 - x - 6) \\ &= 12(x - 3)(x + 2) \end{aligned}$$

Therefore, $f'(x) = 0$ gives $x = -2, 3$. The points $x = -2$ and $x = 3$ divides the real line into three disjoint intervals, namely, $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

Fig 6.4

In the intervals $(-\infty, -2)$ and $(3, \infty)$, $f'(x)$ is positive while in the interval $(-2, 3)$, $f'(x)$ is negative. Consequently, the function f is increasing in the intervals $(-\infty, -2)$ and $(3, \infty)$ while the function is decreasing in the interval $(-2, 3)$. However, f is neither increasing nor decreasing in **R**.

Interval	Sign of $f'(x)$	Nature of function f
$(-\infty, -2)$	$(-)\ (-) > 0$	f is increasing
$(-2, 3)$	$(-)\ (+) < 0$	f is decreasing
$(3, \infty)$	$(+)\ (+) > 0$	f is increasing

Example 12 Find intervals in which the function given by $f(x) = \sin 3x$, $x \in \left[0, \frac{\pi}{2}\right]$ is (a) increasing (b) decreasing.

Solution We have

$$f(x) = \sin 3x$$

or

$$f'(x) = 3\cos 3x$$

Therefore, $f'(x) = 0$ gives $\cos 3x = 0$ which in turn gives $3x = \frac{\pi}{2}, \frac{3\pi}{2}$ (as $x \in \left[0, \frac{\pi}{2}\right]$ implies $3x \in \left[0, \frac{3\pi}{2}\right]$). So $x = \frac{\pi}{6}$ and $\frac{\pi}{2}$. The point $x = \frac{\pi}{6}$ divides the interval $\left[0, \frac{\pi}{2}\right]$ into two disjoint intervals $\left[0, \frac{\pi}{6}\right)$ and $\left(\frac{\pi}{6}, \frac{\pi}{2}\right]$.

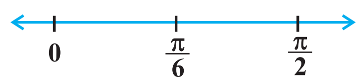


Fig 6.5

Now, $f'(x) > 0$ for all $x \in \left[0, \frac{\pi}{6}\right)$ as $0 \le x < \frac{\pi}{6} \Rightarrow 0 \le 3x < \frac{\pi}{2}$ and $f'(x) < 0$ for all $x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ as $\frac{\pi}{6} < x < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 3x < \frac{3\pi}{2}$.

Therefore, f is increasing in $\left[0, \frac{\pi}{6}\right)$ and decreasing in $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$.

Also, the given function is continuous at $x = 0$ and $x = \frac{\pi}{6}$. Therefore, by Theorem 1,

f is increasing on $\left[0, \frac{\pi}{6}\right]$ and decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Example 13 Find the intervals in which the function f given by

$$f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$$

is increasing or decreasing.

Solution We have

$$f(x) = \sin x + \cos x,$$

or $$f'(x) = \cos x - \sin x$$

Now $f'(x) = 0$ gives $\sin x = \cos x$ which gives that $x = \frac{\pi}{4}, \frac{5\pi}{4}$ as $0 \le x \le 2\pi$

The points $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ divide the interval $[0, 2\pi]$ into three disjoint intervals, namely, $\left[0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$.

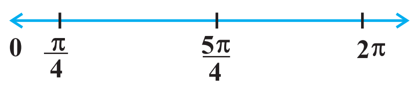


Fig 6.6

Note that $f'(x) > 0$ if $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

or f is increasing in the intervals $\left[0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$

Also $f'(x) < 0$ if $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

or f is decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Interval	Sign of $f'(x)$	Nature of function
$\left[0, \frac{\pi}{4}\right)$	> 0	f is increasing
$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$	< 0	f is decreasing
$\left(\frac{5\pi}{4}, 2\pi\right]$	> 0	f is increasing

EXERCISE 6.2

1. Show that the function given by $f(x) = 3x + 17$ is increasing on **R**.

2. Show that the function given by $f(x) = e^{2x}$ is increasing on **R**.

3. Show that the function given by $f(x) = \sin x$ is

(a) increasing in $\left(0, \frac{\pi}{2}\right)$ (b) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(c) neither increasing nor decreasing in $(0, \pi)$

4. Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is

(a) increasing (b) decreasing

5. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

(a) increasing (b) decreasing

6. Find the intervals in which the following functions are strictly increasing or decreasing:

(a) $x^2 + 2x - 5$ (b) $10 - 6x - 2x^2$

(c) $-2x^3 - 9x^2 - 12x + 1$ (d) $6 - 9x - x^2$

(e) $(x + 1)^3 (x - 3)^3$

7. Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$, is an increasing function of x throughout its domain.

8. Find the values of x for which $y = [x(x-2)]^2$ is an increasing function.

9. Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

10. Prove that the logarithmic function is increasing on $(0, \infty)$.