

15. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1} . Using A^{-1} solve the system of equations

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is ₹70. Find cost of each item per kg by matrix method.

Miscellaneous Examples

Example 30 If a, b, c are positive and unequal, show that value of the determinant

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \text{ is negative.}$$

Solution Applying $C_1 \rightarrow C_1 + C_2 + C_3$ to the given determinant, we get

$$\begin{aligned} \Delta &= \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} \\ &= (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \text{ (Applying } R_2 \rightarrow R_2 - R_1, \text{ and } R_3 \rightarrow R_3 - R_1) \\ &= (a+b+c) [(c-b)(b-c) - (a-c)(a-b)] \text{ (Expanding along } C_1) \\ &= (a+b+c)(-a^2 - b^2 - c^2 + ab + bc + ca) \\ &= \frac{-1}{2} (a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca) \\ &= \frac{-1}{2} (a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] \end{aligned}$$

which is negative (since $a + b + c > 0$ and $(a - b)^2 + (b - c)^2 + (c - a)^2 > 0$)

Example 31 If a, b, c , are in A.P, find value of

$$\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

Solution Applying $R_1 \rightarrow R_1 + R_3 - 2R_2$ to the given determinant, we obtain

$$\begin{vmatrix} 0 & 0 & 0 \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} = 0 \quad (\text{Since } 2b = a + c)$$

Example 32 Show that

$$\Delta = \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$$

Solution Applying $R_1 \rightarrow xR_1, R_2 \rightarrow yR_2, R_3 \rightarrow zR_3$ to Δ and dividing by xyz , we get

$$\Delta = \frac{1}{xyz} \begin{vmatrix} x(y+z)^2 & x^2y & x^2z \\ xy^2 & y(x+z)^2 & y^2z \\ xz^2 & yz^2 & z(x+y)^2 \end{vmatrix}$$

Taking common factors x, y, z from C_1, C_2 and C_3 , respectively, we get

$$\Delta = \frac{xyz}{xyz} \begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$, we have

$$\Delta = \begin{vmatrix} (y+z)^2 & x^2 - (y+z)^2 & x^2 - (y+z)^2 \\ y^2 & (x+z)^2 - y^2 & 0 \\ z^2 & 0 & (x+y)^2 - z^2 \end{vmatrix}$$

Taking common factor $(x + y + z)$ from C_2 and C_3 , we have

$$\Delta = (x + y + z)^2 \begin{vmatrix} (y + z)^2 & x - (y + z) & x - (y + z) \\ y^2 & (x + z) - y & 0 \\ z^2 & 0 & (x + y) - z \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 - (R_2 + R_3)$, we have

$$\Delta = (x + y + z)^2 \begin{vmatrix} 2yz & -2z & -2y \\ y^2 & x - y + z & 0 \\ z^2 & 0 & x + y - z \end{vmatrix}$$

Applying $C_2 \rightarrow (C_2 + \frac{1}{y} C_1)$ and $C_3 \rightarrow C_3 + \frac{1}{z} C_1$, we get

$$\Delta = (x + y + z)^2 \begin{vmatrix} 2yz & 0 & 0 \\ y^2 & x + z & \frac{y^2}{z} \\ z^2 & \frac{z^2}{y} & x + y \end{vmatrix}$$

Finally expanding along R_1 , we have

$$\begin{aligned} \Delta &= (x + y + z)^2 (2yz) [(x + z)(x + y) - yz] = (x + y + z)^2 (2yz) (x^2 + xy + xz) \\ &= (x + y + z)^3 (2xyz) \end{aligned}$$

Example 33 Use product $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 9 & 2 & 3 \\ 6 & 1 & 2 \end{bmatrix}$ to solve the system of equations

$$\begin{aligned} x - y + 2z &= 1 \\ 2y - 3z &= 1 \\ 3x - 2y + 4z &= 2 \end{aligned}$$

Solution Consider the product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

Now, given system of equations can be written, in matrix form, as follows

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

or
$$\begin{matrix} x \\ y \\ z \end{matrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 9 & 2 & 3 \\ 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Hence $x = 0$, $y = 5$ and $z = 3$

Example 34 Prove that

$$\Delta = \begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2) \begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$$

Solution Applying $R_1 \rightarrow R_1 - x R_2$ to Δ , we get

$$\begin{aligned} \Delta &= \begin{vmatrix} a(1-x^2) & c(1-x^2) & p(1-x^2) \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} \\ &= (1-x^2) \begin{vmatrix} a & c & p \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} \end{aligned}$$

Applying $R_2 \rightarrow R_2 - x R_1$, we get

$$\Delta = (1-x^2) \begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$$

Miscellaneous Exercises on Chapter 4

1. Prove that the determinant $\begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$ is independent of θ .

2. Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

3. Evaluate $\begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix}$.

4. If a , b and c are real numbers, and

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0,$$

Show that either $a + b + c = 0$ or $a = b = c$.

5. Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$

6. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

7. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$