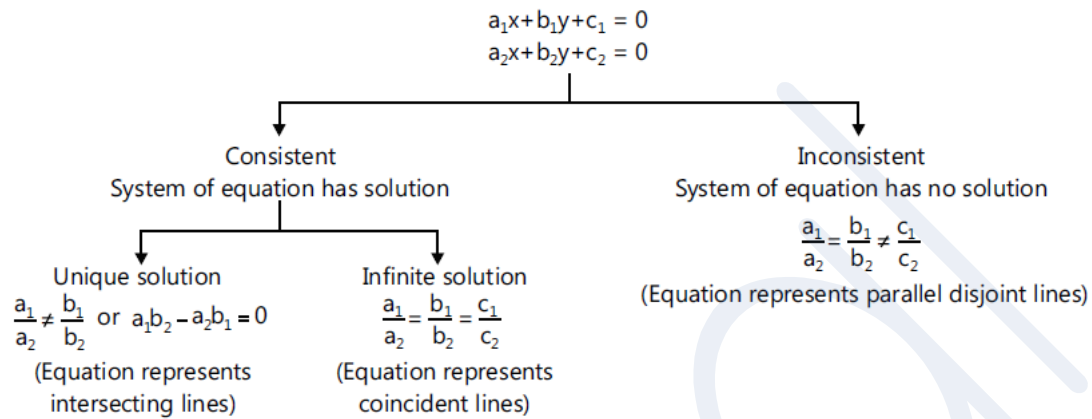# 1 Consistency and Inconsistency

A system of linear equations having two and three variables can be easily solved using determinants. When system of equations has at least one solution it is called consistent, and if there are no solutions at all then it is called inconsistent. Below diagram clearifies,

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

Consistent
System of equation has solution

Inconsistent
System of equation has no solution
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$
(Equation represents parallel disjoint lines)

Unique solution
$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \text{ or } a_1b_2 - a_2b_1 = 0$$
(Equation represents intersecting lines)

Infinite solution
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$
(Equation represents coincident lines)

2 Cramer's Rule

For 3 linear equations system,

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_1$$
$$a_3x + b_3y + c_3z = d_1$$

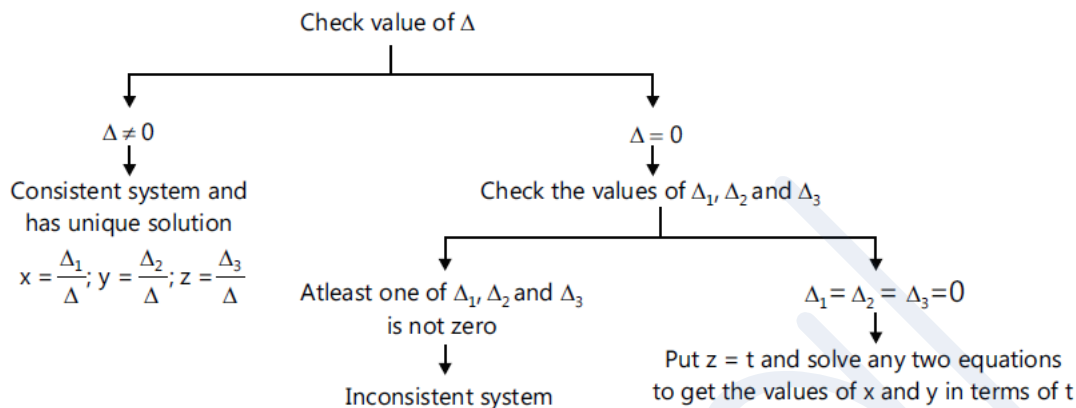Define following determinants,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Now Cramer's rule gives solution when $\Delta \neq 0$

$$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

3 Conditions for Infinite and No Solutions

Check value of Δ

$\Delta \neq 0$

Consistent system and has unique solution

$x = \dfrac{\Delta_1}{\Delta}$; $y = \dfrac{\Delta_2}{\Delta}$; $z = \dfrac{\Delta_3}{\Delta}$

$\Delta = 0$

Check the values of Δ_1, Δ_2 and Δ_3

Atleast one of Δ_1, Δ_2 and Δ_3 is not zero

Inconsistent system

$\Delta_1 = \Delta_2 = \Delta_3 = 0$

Put z = t and solve any two equations to get the values of x and y in terms of t

Points

1. If $\Delta \neq 0$ then the given system of equations are consistent and have unique solution.

2. If $\Delta = 0$ but at least one of $\Delta_1, \Delta_2, \Delta_3$ is not zero then the equations are inconsistent and have no solution.

3. If $\Delta_1 = \Delta_2 = \Delta_3 = \Delta = 0$ then the given system of equations are consistent and have infinite solution except the case of parallel planes when there is no solution.

4 Homogeneous Linear Equations

If $d_1 = d_2 = d_3 = 0$, then system of linear equations is known as Homogeneous linear equations, which always possess at least one solution i.e. (0, 0, 0). This is called a trivial solution for homogeneous linear equations.

1. Solution of Homogenous Equations is always consistent, as $x = 0 = y = z$ is always a solution. This is known as TRIVIAL solution.

2. For Homogenous Equations, if $\Delta \neq 0$. Then $x = 0 = y = z$ is only solution.

3. For Homogenous Equations, if $\Delta = 0$, then there exists non zero solutions (**NON TRIVIAL SOLUTIONS**) also.