Question 1: (JEE Main 2019)

Let $f(x)=\int_0^x g(t)dt$, where g is a non-zero even function. If $f(x+5)=g(x)$, then $\int_0^x f(t)dt$ equals-

(1) $\int_{x+5}^{5} g(t)dt$ (2) $5\int_{x+5}^{5} g(t)dt$

(3) $\int_{5}^{x+5} g(t)dt$ (4) $2\int_{5}^{x+5} g(t)dt$

Sol:

$$f(x)=\int_0^x g(t)dt$$

$$f(-x)=\int_0^{-x} g(t)dt$$

put $t=-u$

$$=-\int_0^x g(-u)du$$

$$=-\int_0^x g(u)d(u)=-f(x)$$

$\Rightarrow f(-x)=-f(x)$

$\Rightarrow f(x)$ is an odd function

Also $f(5+x)=g(x)$

$f(5-x)=g(-x)=g(x)=f(5+x)$

$\Rightarrow f(5-x)=f(5+x)$

Now

$$I=\int_0^x f(t)dt$$

$t=u+5$

$$I=\int_{-5}^{x-5} f(u+5)du$$

$$=\int_{-5}^{x-5} g(u)du$$

$$=\int_{-5}^{x-5} f'(u)du$$

$=f(x-5)-f(-5)$

$=-f(5-x)+f(5)$

$=f(5)-f(5+x)$

$$=\int_{5+x}^{5} f'(t)dt=\int_{5+x}^{5} g(t)dt$$

Question 2: (JEE Main 2020)

$\int_{-\pi}^{\pi} \left| \pi - |x| \right| dx$ is equal to :

(1) π^2 (2) $2\pi^2$

(3) $\sqrt{2}\pi^2$ (4) $\frac{\pi^2}{2}$

Sol:

$$\int_{-\pi}^{\pi} \left|\pi - |x|\right| dx = 2\int_{0}^{\pi} |\pi - x| dx$$

$$= 2\int_{0}^{\pi} (\pi - x)\, dx$$

$$= 2\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \pi^2$$

Question 3: (JEE Advanced 2021)

For any real number x, let [x] denote the largest integer less than or equal to x. If

$$I = \int_0^{10} \left[\sqrt{\frac{10x}{x+1}}\right] dx,$$

then the value of 9I is ____.

Solution:

Let $f(x) = \left(\frac{10x}{x+1}\right)$

So, $f'(x) = 10\left(\frac{(x+1)-x}{(x+1)^2}\right) = \frac{10}{(x+1)^2} > 0 \ \forall\ x \in [0, 10]$,

So, $f(x)$ is an increasing function

So, range of $f(x)$ is $\left[0, \sqrt{\frac{100}{11}}\right]$

$$I = \int_0^{1/9}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{1/9}^{2/3}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{9}^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$$

$$= 0 + \int_{1/9}^{2/3} dx + 2\int_{2/3}^{9} dx + 3\int_{9}^{10} dx$$

$$= \frac{2}{3} - \frac{1}{9} + 2\left(9 - \frac{2}{3}\right) + 3(10 - 9)$$

$$= \frac{6-1}{9} + 2 \times \frac{25}{3} + 3 = \frac{5}{9} + \frac{50}{3} + 3$$

$$= \frac{5 + 150 + 27}{9} = \frac{182}{9} = 182$$

Question 4: (JEE Advanced 2021)

Let $g_i : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R$, $i = 1, 2$ and $f : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R$ be functions such that

$g_1(x) = 1$, $g_2(x) = |4x - \pi|$ and $f(x) = \sin^2 x$, for all $x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$. Define $S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x) \cdot g_i(x) dx,\ i = 1, 2$

a.

The value of $\frac{16 S_1}{\pi}$ is _____

Sol:

$$g_1 : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R,\ i = 1,2,\ f : \left[\frac{\pi}{r}, \frac{3\pi}{r}\right] \to R$$

$g_1 = 1$, $g_2 = |4x - \pi|$, $f(x) = \sin^2 x$

$$S_i = \int_{\pi/8}^{3\pi/8} f(x) \cdot g_i(x)\, dx$$

$$S_1 = \int_{\pi/8}^{3\pi/8} \sin^2 x\, dx = \int_{\pi/8}^{3\pi/8} \sin^2\left(\frac{\pi}{2} - x\right) dx \Rightarrow 2S_1 = \int_{\pi/8}^{3\pi/8} 1\, dx$$

$$\Rightarrow S_1 = \frac{1}{2}\left(\frac{3\pi}{8} - \frac{\pi}{8}\right) = \frac{\pi}{8} \Rightarrow \frac{16S_1}{\pi} = 2$$

b.

The value of $\frac{48 S_2}{\pi^2}$ is _____

Sol:

$$S_2 = \int_{\pi/8}^{3\pi/8} \sin^2 x |4x - \pi| dx = \int_{\pi/8}^{3\pi/8} \sin^2\left(\frac{\pi}{2} - x\right)\left|4\left(\frac{\pi}{2} - x\right) - \pi\right| dx$$

$$= \int_{\pi/8}^{3\pi/8} \cos^2 x |4x - \pi|\, dx$$

$$\Rightarrow 2S_2 = \int_{\pi/8}^{3\pi/8} |4x - \pi| dx = 2\int_{\pi/8}^{\pi/4} (\pi - 4x)\, dx$$

$$S_2 = \left(\pi x - 2x^2\right)\Big|_{\pi/8}^{\pi/4} = \pi\left(\frac{\pi}{4} - \frac{\pi}{8}\right) - 2\left(\frac{\pi^2}{16} - \frac{\pi^2}{64}\right)$$

$$S_2 = \frac{4\pi^2}{8} - \frac{3\pi^2}{32} = \frac{\pi^2}{32} = \frac{48S_2}{\pi^2} = \frac{48}{\pi^2} \times \frac{\pi^2}{32} = \frac{3}{2}$$

$$\frac{48S_2}{\pi^2} = 1.5$$

Question 5: (JEE Advanced 2018)

Let $f : R \to R$ and $g : R \to R$ be two non-constant differentiable functions. If

$$f'(x) = \left(e^{(f(x)-g(x))}\right) g'(x) \text{ for all } x \in R,$$

and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE ?

(A) $f(2) < 1 - \log_e 2$ (B) $f(2) > 1 - \log_e 2$

(C) $g(1) > 1 - \log_e 2$ (D) $g(1) < 1 - \log_e 2$

Solution:

$$f'(x) = e^{f(x)-g(x)} \cdot g'(x) \text{ given } f(1) = g(2) = 1$$

$$\int -f'(x)\cdot e^{-f(x)}dx = \int -e^{-g(x)}\cdot g'(x)\,dx$$

$$\int d\left(e^{-f(x)}\right) = \int d\left(e^{-g(x)}\right) + C$$

$$e^{-f(x)} = e^{-g(x)} + C$$

put $x = 1 \Rightarrow C = \dfrac{1}{e} - \dfrac{1}{e^{g(1)}}$

$$x = 2 \Rightarrow e^{f(2)} = \frac{e^{1+g(1)}}{2e^{g(1)} - e}$$

$\Rightarrow$ $g(1) > 1 - \log_e 2$ & $f(2) > 1 - \log_e 2$

Question 6: (JEE Adavanced 2018)

The value of the integral $\displaystyle\int_0^{1/2} \frac{1+\sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{1/4}}dx$ is ______ .

Solution:

$$I_2 = \int_0^{1/2} \frac{1+\sqrt{3}}{\sqrt{1-x^2}\,(1-x)}dx$$

Let $x = \sin\theta$

$$I_2 = \int_0^{\pi/6} \frac{1+\sqrt{3}}{\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^2}d\theta = \int_0^{\pi/6} \frac{\left(1+\sqrt{3}\right)\sec^2\frac{\theta}{2}}{\left(\tan\frac{\theta}{2} - 1\right)^2}d\theta$$

Let $\tan\dfrac{\theta}{2} = y$

$$\Rightarrow I = \int_0^{2-\sqrt{3}} \frac{2\left(1+\sqrt{3}\right)dy}{(y-1)^2} = 2\left(1+\sqrt{3}\right)\left|\frac{-1}{y-1}\right|_0^{2-\sqrt{3}} = 2\left(1+\sqrt{3}\right)\left|\frac{-1}{1-\sqrt{3}} - 1\right| = 2\left(1+\sqrt{3}\right)\left(\frac{\sqrt{3}-1}{2}\right) = 2$$

Question 7: (JEE Advanced 2016)

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x} dx$ is equal to

(A) $\frac{\pi^2}{4} - 2$

(B) $\frac{\pi^2}{4} + 2$

(C) $\pi^2 - e^{\frac{\pi}{2}}$

(D) $\pi^2 + e^{\frac{\pi}{2}}$

Solution:

$$= \int_0^{\pi/2} \left(\frac{x^2 \cos x}{1+e^x} + \frac{x^2 \cos x}{1+e^{-x}} \right) dx$$

$$= \int_0^{\pi/2} \frac{x^2 \cos x + x^2 e^x \cos x}{1+e^x} dx$$

$$= \int_0^{\pi/2} x^2 \cos x \, dx$$

$$= \left(x^2 \sin x\right)_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x dx$$

$$= \frac{\pi^2}{4} - 2\left[\left[\left(x(-\cos x) \right) \right]_0^{\pi/2} - \int_0^{\pi/2} -\cos x dx \right]$$

$$= \frac{\pi^2}{4} - 2\left[-(0-0) + (\sin x) \right]_0^{\pi/2}$$

$$= \frac{\pi^2}{4} - 2$$

Question 8: (JEE Advanced 2015)

If

$$\alpha = \int_0^1 \left(e^{9x+3\tan^{-1} x} \right) \left(\frac{12+9x^2}{1+x^2} \right) dx$$

where $\tan^{-1} x$ takes only principal values, then the value of $\left(\log_e |1+\alpha| - \frac{3\pi}{4} \right)$ is

Solution:

$$\alpha = \int_0^1 e^{\left(9x+3\tan^{-1}x\right)}\left(\frac{12+9x^2}{1+x^2}\right)dx$$

Put $9x + 3\tan^{-1} x = t$

$$\Rightarrow \left(9+\frac{3}{1+x^2}\right)dx = dt$$

$$\Rightarrow \alpha = \int_0^{9+\frac{3\pi}{4}} e^t dt = e^{9+\frac{3\pi}{4}} - 1$$

$$\Rightarrow \left(\log_e |1+\alpha| - \frac{3\pi}{4}\right) = 9$$