## EXERCISE 7.9

Evaluate the definite integrals in Exercises 1 to 20.

**1.** $\int_{-1}^{1}(x+1)\,dx$ **2.** $\int_{2}^{3}\frac{1}{x}\,dx$ **3.** $\int_{1}^{2}(4x^3-5x^2+6x+9)\,dx$

**4.** $\int_{0}^{\frac{\pi}{4}}\sin 2x\,dx$ **5.** $\int_{0}^{\frac{\pi}{2}}\cos 2x\,dx$ **6.** $\int_{4}^{5}e^x\,dx$ **7.** $\int_{0}^{\frac{\pi}{4}}\tan x\,dx$

**8.** $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\operatorname{cosec} x\,dx$ **9.** $\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}}$ **10.** $\int_{0}^{1}\frac{dx}{1+x^2}$ **11.** $\int_{2}^{3}\frac{dx}{x^2-1}$

**12.** $\int_{0}^{\frac{\pi}{2}}\cos^2 x\,dx$ **13.** $\int_{2}^{3}\frac{x\,dx}{x^2+1}$ **14.** $\int_{0}^{1}\frac{2x+3}{5x^2+1}\,dx$ **15.** $\int_{0}^{1}x\,e^{x^2}\,dx$

**16.** $\int_{1}^{2}\frac{5x^2}{x^2+4x+3}$ **17.** $\int_{0}^{\frac{\pi}{4}}(2\sec^2 x+x^3+2)\,dx$ **18.** $\int_{0}^{\pi}(\sin^2\frac{x}{2}-\cos^2\frac{x}{2})\,dx$

**19.** $\int_{0}^{2}\frac{6x+3}{x^2+4}\,dx$ **20.** $\int_{0}^{1}(x\,e^x+\sin\frac{\pi x}{4})\,dx$

Choose the correct answer in Exercises 21 and 22.

**21.** $\int_{1}^{\sqrt{3}}\frac{dx}{1+x^2}$ equals

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

**22.** $\int_{0}^{\frac{2}{3}}\frac{dx}{4+9x^2}$ equals

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{12}$ (C) $\frac{\pi}{24}$ (D) $\frac{\pi}{4}$

## 7.9 Evaluation of Definite Integrals by Substitution

In the previous sections, we have discussed several methods for finding the indefinite integral. One of the important methods for finding the indefinite integral is the method of substitution.

To evaluate $\int_a^b f(x)\,dx$, by substitution, the steps could be as follows:

1. Consider the integral without limits and substitute, $y = f(x)$ or $x = g(y)$ to reduce the given integral to a known form.
2. Integrate the new integrand with respect to the new variable without mentioning the constant of integration.
3. Resubstitute for the new variable and write the answer in terms of the original variable.
4. Find the values of answers obtained in (3) at the given limits of integral and find the difference of the values at the upper and lower limits.

**Note** In order to quicken this method, we can proceed as follows: After performing steps 1, and 2, there is no need of step 3. Here, the integral will be kept in the new variable itself, and the limits of the integral will accordingly be changed, so that we can perform the last step.

Let us illustrate this by examples.

**Example 28** Evaluate $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx$.

**Solution** Put $t = x^5 + 1$, then $dt = 5x^4\,dx$.

Therefore, $$\int 5x^4\sqrt{x^5+1}\,dx = \int \sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}(x^5+1)^{\frac{3}{2}}$$

Hence, $$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \frac{2}{3}\left[(x^5+1)^{\frac{3}{2}}\right]_{-1}^{1}$$

$$= \frac{2}{3}\left[(1^5+1)^{\frac{3}{2}} - \left((-1)^5+1\right)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

**Alternatively**, first we transform the integral and then evaluate the transformed integral with new limits.

Let $t = x^5 + 1$. Then $dt = 5\,x^4\,dx$.

Note that, when $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

Thus, as $x$ varies from – 1 to 1, $t$ varies from 0 to 2

Therefore $$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \int_0^2 \sqrt{t}\,dt$$

$$= \frac{2}{3}\left[t^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

**Example 29** Evaluate $\int_0^1 \frac{\tan^{-1} x}{1+x^2}\,dx$

**Solution** Let $t = \tan^{-1}x$, then $dt = \frac{1}{1+x^2}\,dx$. The new limits are, when $x = 0, t = 0$ and when $x = 1$, $t = \frac{\pi}{4}$. Thus, as $x$ varies from 0 to 1, $t$ varies from 0 to $\frac{\pi}{4}$.

Therefore $$\int_0^1 \frac{\tan^{-1} x}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} t\,dt \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = \frac{\pi^2}{32}$$

## EXERCISE 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution.

**1.** $\int_0^1 \frac{x}{x^2+1}\,dx$ **2.** $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi$ **3.** $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

**4.** $\int_0^2 x\sqrt{x+2}$ (Put $x + 2 = t^2$) **5.** $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$

**6.** $\int_0^2 \frac{dx}{x+4-x^2}$ **7.** $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$ **8.** $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right)e^{2x}\,dx$

Choose the correct answer in Exercises 9 and 10.

**9.** The value of the integral $\int_{\frac{1}{3}}^{1} \frac{(x-x^3)^{\frac{1}{3}}}{x^4}\,dx$ is

(A) 6 (B) 0 (C) 3 (D) 4

**10.** If $f(x) = \int_0^x t\sin t\,dt$, then $f'(x)$ is

(A) $\cos x + x\sin x$ (B) $x\sin x$

(C) $x\cos x$ (D) $\sin x + x\cos x$