7.3 Methods of Integration

In previous section, we discussed integrals of those functions which were readily obtainable from derivatives of some functions. It was based on inspection, i.e., on the search of a function F whose derivative is f which led us to the integral of f. However, this method, which depends on inspection, is not very suitable for many functions. Hence, we need to develop additional techniques or methods for finding the integrals by reducing them into standard forms. Prominent among them are methods based on:

1. Integration by Substitution
2. Integration using Partial Fractions
3. Integration by Parts

7.3.1 *Integration by substitution*

In this section, we consider the method of integration by substitution.

The given integral $\int f(x)\, dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$.

Consider $\qquad \mathrm{I} = \int f(x)\, dx$

Put $x = g(t)$ so that $\dfrac{dx}{dt} = g'(t)$.

We write $\qquad dx = g'(t)\, dt$

Thus $\qquad \mathrm{I} = \int f(x)\, dx = \int f(g(t))\, g'(t)\, dt$

This change of variable formula is one of the important tools available to us in the name of integration by substitution. It is often important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand as illustrated in the following examples.

Example 5 Integrate the following functions w.r.t. x:

(i) $\sin mx$ $\qquad$ (ii) $2x \sin (x^2 + 1)$

(iii) $\dfrac{\tan^4 \sqrt{x}\, \sec^2 \sqrt{x}}{\sqrt{x}}$ $\qquad$ (iv) $\dfrac{\sin (\tan^{-1} x)}{1+x^2}$

Solution

(i) We know that derivative of mx is m. Thus, we make the substitution $mx = t$ so that $mdx = dt$.

Therefore, $\quad \int \sin mx\, dx = \dfrac{1}{m}\int \sin t\, dt = -\dfrac{1}{m} \cos t + \mathrm{C} = -\dfrac{1}{m} \cos mx + \mathrm{C}$

(ii) Derivative of $x^2 + 1$ is $2x$. Thus, we use the substitution $x^2 + 1 = t$ so that $2x\,dx = dt$.

Therefore, $\int 2x \sin(x^2+1)\,dx = \int \sin t\,dt = -\cos t + C = -\cos(x^2+1) + C$

(iii) Derivative of $\sqrt{x}$ is $\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. Thus, we use the substitution

$\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}}\,dx = dt$ giving $dx = 2t\,dt$.

Thus, $$\int \frac{\tan^4\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{2t\tan^4 t\sec^2 t\,dt}{t} = 2\int \tan^4 t\ \sec^2 t\,dt$$

Again, we make another substitution $\tan t = u$ so that $\sec^2 t\,dt = du$

Therefore, $$2\int \tan^4 t\sec^2 t\,dt = 2\int u^4\,du = 2\frac{u^5}{5} + C$$

$$= \frac{2}{5}\tan^5 t + C \text{ (since } u = \tan t)$$

$$= \frac{2}{5}\tan^5\sqrt{x} + C \text{ (since } t = \sqrt{x})$$

Hence, $$\int \frac{\tan^4\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}\,dx = \frac{2}{5}\tan^5\sqrt{x} + C$$

Alternatively, make the substitution $\tan\sqrt{x} = t$

(iv) Derivative of $\tan^{-1}x = \frac{1}{1+x^2}$. Thus, we use the substitution

$$\tan^{-1}x = t \text{ so that } \frac{dx}{1+x^2} = dt.$$

Therefore, $$\int \frac{\sin(\tan^{-1}x)}{1+x^2}\,dx = \int \sin t\,dt = -\cos t + C = -\cos(\tan^{-1}x) + C$$

Now, we discuss some important integrals involving trigonometric functions and their standard integrals using substitution technique. These will be used later without reference.

(i) $\int \tan x\,dx = \log|\sec x| + C$

We have

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

Put $\cos x = t$ so that $\sin x \, dx = -dt$

Then $$\int \tan x \, dx = -\int \frac{dt}{t} = -\log|t| + C = -\log|\cos x| + C$$

or $$\int \tan x \, dx = \log|\sec x| + C$$

(ii) $\int \cot x \, dx = \log|\sin x| + C$

We have $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$

Put $\sin x = t$ so that $\cos x \, dx = dt$

Then $$\int \cot x \, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sin x| + C$$

(iii) $\int \sec x \, dx = \log|\sec x + \tan x| + C$

We have

$$\int \sec x \, dx = \int \frac{\sec x \, (\sec x + \tan x)}{\sec x + \tan x} \, dx$$

Put $\sec x + \tan x = t$ so that $\sec x \, (\tan x + \sec x) \, dx = dt$

Therefore, $\int \sec x \, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sec x + \tan x| + C$

(iv) $\int \text{cosec}\, x \, dx = \log|\text{cosec}\, x - \cot x| + C$

We have

$$\int \text{cosec}\, x \, dx = \int \frac{\text{cosec}\, x \, (\text{cosec}\, x + \cot x)}{(\text{cosec}\, x + \cot x)} \, dx$$

Put $\text{cosec}\, x + \cot x = t$ so that $-\text{cosec}\, x \, (\text{cosec}\, x + \cot x) \, dx = dt$

So $$\int \text{cosec}\, x \, dx = -\int \frac{dt}{t} = -\log|t| = -\log|\text{cosec}\, x + \cot x| + C$$

$$= -\log\left|\frac{\text{cosec}^2 x - \cot^2 x}{\text{cosec}\, x - \cot x}\right| + C$$

$$= \log|\text{cosec}\, x - \cot x| + C$$

Example 6 Find the following integrals:

(i) $\int \sin^3 x \cos^2 x \, dx$ (ii) $\int \frac{\sin x}{\sin (x + a)} \, dx$ (iii) $\int \frac{1}{1 + \tan x} \, dx$

Solution

(i) We have

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \, (\sin x) \, dx$$

$$= \int (1 - \cos^2 x) \cos^2 x \, (\sin x) \, dx$$

Put $t = \cos x$ so that $dt = -\sin x \, dx$

Therefore, $\int \sin^2 x \cos^2 x \, (\sin x) \, dx = -\int (1 - t^2) \, t^2 \, dt$

$$= -\int (t^2 - t^4) \, dt = -\left(\frac{t^3}{3} - \frac{t^5}{5}\right) + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

(ii) Put $x + a = t$. Then $dx = dt$. Therefore

$$\int \frac{\sin x}{\sin (x + a)} \, dx = \int \frac{\sin (t - a)}{\sin t} \, dt$$

$$= \int \frac{\sin t \cos a - \cos t \sin a}{\sin t} \, dt$$

$$= \cos a \int dt - \sin a \int \cot t \, dt$$

$$= (\cos a) \, t - (\sin a) \left[\log |\sin t| + C_1\right]$$

$$= (\cos a) \, (x + a) - (\sin a) \left[\log |\sin (x + a)| + C_1\right]$$

$$= x \cos a + a \cos a - (\sin a) \log |\sin (x + a)| - C_1 \sin a$$

Hence, $\int \frac{\sin x}{\sin (x + a)} \, dx = x \cos a - \sin a \log |\sin (x + a)| + C,$

where, $C = -C_1 \sin a + a \cos a$, is another arbitrary constant.

(iii) $\int \frac{dx}{1 + \tan x} = \int \frac{\cos x \, dx}{\cos x + \sin x}$

$$= \frac{1}{2} \int \frac{(\cos x + \sin x + \cos x - \sin x) \, dx}{\cos x + \sin x}$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx$$

$$= \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx \qquad \text{... (1)}$$

Now, consider $\mathrm{I} = \int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx$

Put $\cos x + \sin x = t$ so that $(\cos x - \sin x)\, dx = dt$

Therefore $\mathrm{I} = \int \frac{dt}{t} = \log|t| + C_2 = \log|\cos x + \sin x| + C_2$

Putting it in (1), we get

$$\int \frac{dx}{1 + \tan x} = \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{C_2}{2}$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{C_1}{2} + \frac{C_2}{2}$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + C, \left(C = \frac{C_1}{2} + \frac{C_2}{2}\right)$$

EXERCISE 7.2

Integrate the functions in Exercises 1 to 37:

1. $\frac{2x}{1+x^2}$ **2.** $\frac{(\log x)^2}{x}$ **3.** $\frac{1}{x + x\log x}$

4. $\sin x \sin(\cos x)$ **5.** $\sin(ax+b)\cos(ax+b)$

6. $\sqrt{ax+b}$ **7.** $x\sqrt{x+2}$ **8.** $x\sqrt{1+2x^2}$

9. $(4x+2)\sqrt{x^2+x+1}$ **10.** $\frac{1}{x-\sqrt{x}}$ **11.** $\frac{x}{\sqrt{x+4}}, x > 0$

12. $(x^3-1)^{\frac{1}{3}}x^5$ **13.** $\frac{x^2}{(2+3x^3)^3}$ **14.** $\frac{1}{x(\log x)^m}, x > 0, m \neq 1$

15. $\frac{x}{9-4x^2}$ **16.** e^{2x+3} **17.** $\frac{x}{e^{x^2}}$

18. $\dfrac{e^{\tan^{-1}x}}{1+x^2}$ **19.** $\dfrac{e^{2x}-1}{e^{2x}+1}$ **20.** $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

21. $\tan^2(2x-3)$ **22.** $\sec^2(7-4x)$ **23.** $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

24. $\dfrac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$ **25.** $\dfrac{1}{\cos^2 x(1-\tan x)^2}$ **26.** $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$

27. $\sqrt{\sin 2x}\cos 2x$ **28.** $\dfrac{\cos x}{\sqrt{1+\sin x}}$ **29.** $\cot x \log \sin x$

30. $\dfrac{\sin x}{1+\cos x}$ **31.** $\dfrac{\sin x}{(1+\cos x)^2}$ **32.** $\dfrac{1}{1+\cot x}$

33. $\dfrac{1}{1-\tan x}$ **34.** $\dfrac{\sqrt{\tan x}}{\sin x\cos x}$ **35.** $\dfrac{(1+\log x)^2}{x}$

36. $\dfrac{(x+1)(x+\log x)^2}{x}$ **37.** $\dfrac{x^3\sin(\tan^{-1}x^4)}{1+x^8}$

Choose the correct answer in Exercises 38 and 39.

38. $\displaystyle\int \frac{10x^9 + 10^x \log_{e^{10}} dx}{x^{10}+10^x}$ equals

(A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$

(C) $(10^x - x^{10})^{-1} + C$ (D) $\log(10^x + x^{10}) + C$

39. $\displaystyle\int \frac{dx}{\sin^2 x\cos^2 x}$ equals

(A) $\tan x + \cot x + C$ (B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$ (D) $\tan x - \cot 2x + C$

7.3.2 *Integration using trigonometric identities*

When the integrand involves some trigonometric functions, we use some known identities to find the integral as illustrated through the following example.

Example 7 Find (i) $\int \cos^2 x\,dx$ (ii) $\int \sin 2x \cos 3x\,dx$ (iii) $\int \sin^3 x\,dx$