Now, let us consider the motion of a car that starts from rest at time $t = 0$ s from the origin O and picks up speed till $t = 10$ s and thereafter moves with uniform speed till $t = 18$ s. Then the brakes are applied and the car stops at $t = 20$ s and $x = 296$ m. The position-time graph for this case is shown in Fig. 3.3. We shall refer to this graph in our discussion in the following sections.

3.3 AVERAGE VELOCITY AND AVERAGE SPEED

When an object is in motion, its position changes with time. But how fast is the position changing with time and in what direction? To describe this, we define the quantity **average velocity**. Average velocity is defined as the change in position or displacement (Δx) divided by the time intervals (Δt), in which the displacement occurs :

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \tag{3.1}$$

where x_2 and x_1 are the positions of the object at time t_2 and t_1, respectively. Here the bar over the symbol for velocity is a standard notation used to indicate an average quantity. The SI unit for velocity is m/s or m s^{-1}, although km h^{-1} is used in many everyday applications.

Like displacement, average velocity is also a vector quantity. But as explained earlier, for motion in a straight line, the directional aspect of the vector can be taken care of by + and – signs and we do not have to use the vector notation for velocity in this chapter.

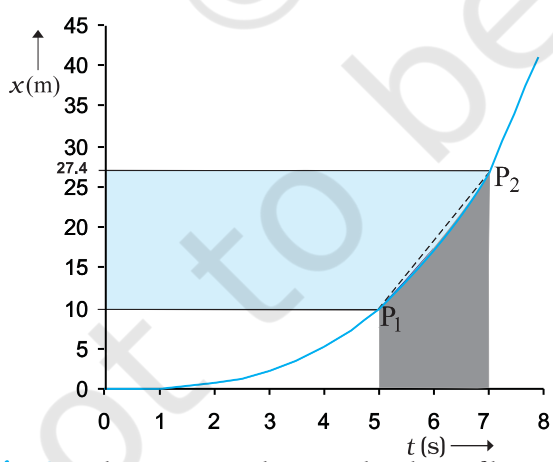


Fig. 3.4 The average velocity is the slope of line P_1P_2.

Consider the motion of the car in Fig. 3.3. The portion of the x-t graph between $t = 0$ s and $t = 8$ s is blown up and shown in Fig. 3.4. As seen from the plot, the average velocity of the car between time $t = 5$ s and $t = 7$ s is :

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{(27.4 - 10.0)\,\text{m}}{(7 - 5)\,\text{s}} = 8.7\ \text{m s}^{-1}$$

Geometrically, this is the slope of the straight line P_1P_2 connecting the initial position P_1 to the final position P_2 as shown in Fig. 3.4.

The average velocity can be positive or negative depending upon the sign of the displacement. It is zero if the displacement is zero. Fig. 3.5 shows the x-t graphs for an object, moving with positive velocity (Fig. 3.5a), moving with negative velocity (Fig. 3.5b) and at rest (Fig. 3.5c).

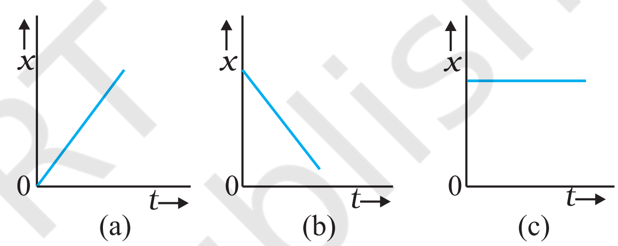


Fig. 3.5 *Position-time graph for an object (a) moving with positive velocity, (b) moving with negative velocity, and (c) at rest.*

Average velocity as defined above involves only the displacement of the object. We have seen earlier that the magnitude of displacement may be different from the actual path length. To describe the rate of motion over the actual path, we introduce another quantity called **average speed**.

Average speed is defined as the total path length travelled divided by the total time interval during which the motion has taken place :

$$\textit{Average speed} = \frac{\textit{Total path length}}{\textit{Total time interval}} \tag{3.2}$$

Average speed has obviously the same unit (m s^{-1}) as that of velocity. But it does not tell us in what direction an object is moving. Thus, it is always positive (in contrast to the average velocity which can be positive or negative). If the motion of an object is along a straight line and in the **same direction**, the magnitude of displacement is equal to the total path length. In that case, the magnitude of average velocity is equal to the average speed. This is not always the case, as you will see in the following example.

▶ ***Example 3.1*** A car is moving along a straight line, say OP in Fig. 3.1. It moves from O to P in 18 s and returns from P to Q in 6.0 s. What are the average velocity and average speed of the car in going (a) from O to P ? and (b) from O to P and back to Q ?

Answer (a)

$$\text{Average velocity} = \frac{\text{Displacement}}{\text{Time interval}}$$

$$\bar{v} = \frac{+\,360\text{ m}}{18\text{ s}} = +\,20\text{ m s}^{-1}$$

$$\text{Average speed} = \frac{\text{Path length}}{\text{Time interval}}$$

$$= \frac{360\text{ m}}{18\text{ s}} = 20\text{ m s}^{-1}$$

Thus, in this case the average speed is equal to the magnitude of the average velocity.

(b) In this case,

$$\text{Average velocity} = \frac{\text{Displacement}}{\text{Time interval}} = \frac{+240\text{ m}}{(18+6.0)\text{ s}}$$

$$= +10\text{ m s}^{-1}$$

$$\text{Average speed} = \frac{\text{Path length}}{\text{Time interval}} = \frac{\text{OP + PQ}}{\Delta t}$$

$$= \frac{(360+120)\text{ m}}{24\text{ s}} = 20\text{ m s}^{-1}$$

Thus, in this case the average speed is *not* equal to the magnitude of the average velocity. This happens because the motion here involves change in direction so that the path length is greater than the magnitude of displacement. This shows that **speed is, in general, greater than the magnitude of the velocity.** ◀

If the car in Example 3.1 moves from O to P and comes back to O in the same time interval, average speed is 20 m/s but the average velocity is zero !

3.4 INSTANTANEOUS VELOCITY AND SPEED

The average velocity tells us how fast an object has been moving over a given time interval but does not tell us how fast it moves at different instants of time during that interval. For this, we define **instantaneous velocity** or simply velocity v at an instant t.

The velocity at an instant is defined as the limit of the average velocity as the time interval Δt becomes infinitesimally small. In other words,

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \qquad (3.3a)$$

$$= \frac{dx}{dt} \qquad (3.3b)$$

where the symbol $\lim_{\Delta t \to 0}$ stands for the operation of taking limit as $\Delta t \to 0$ of the quantity on its right. In the language of calculus, the quantity on the right hand side of Eq. (3.3a) is the differential coefficient of x with respect to t and is denoted by $\frac{dx}{dt}$ (see Appendix 3.1). It is the rate of change of position with respect to time, at that instant.

We can use Eq. (3.3a) for obtaining the value of velocity at an instant either **graphically** or **numerically**. Suppose that we want to obtain graphically the value of velocity at time $t = 4$ s (point P) for the motion of the car represented in Fig. 3.3. The figure has been redrawn in Fig. 3.6 choosing different scales to facilitate the

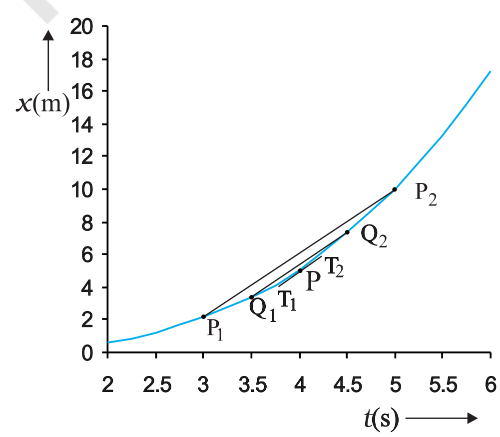


Fig. 3.6 *Determining velocity from position-time graph. Velocity at t = 4 s is the slope of the tangent to the graph at that instant.*

calculation. Let us take Δt = 2 s centred at t = 4 s. Then, by the definition of the average velocity, the slope of line P_1P_2 (Fig. 3.6) gives the value of average velocity over the interval 3 s to 5 s. Now, we decrease the value of Δt from 2 s to 1 s. Then line P_1P_2 becomes Q_1Q_2 and its slope gives the value of the average velocity over the interval 3.5 s to 4.5 s. In the limit $\Delta t \to 0$, the line P_1P_2 becomes tangent to the position-time curve at the point P and the velocity at t = 4 s is given by the slope of the tangent at that point. **It is difficult to show this process graphically. But if we use numerical method to obtain the value of the velocity, the meaning of the limiting process becomes clear**. For the graph shown in Fig. 3.6, $x = 0.08\ t^3$. Table 3.1 gives the value of $\Delta x/\Delta t$ calculated for Δt equal to 2.0 s, 1.0 s, 0.5 s, 0.1 s and 0.01 s centred at t = 4.0 s. The second and third columns give the value of $t_1 = \left(t - \frac{\Delta t}{2}\right)$ and $t_2 = \left(t + \frac{\Delta t}{2}\right)$ and the fourth and the fifth columns give the corresponding values of x, i.e. $x(t_1) = 0.08\ t_1^3$ and $x(t_2) = 0.08\ t_2^3$. The sixth column lists the difference $\Delta x = x(t_2) - x(t_1)$ and the last column gives the ratio of Δx and Δt, i.e. the average velocity corresponding to the value of Δt listed in the first column.

We see from Table 3.1 that as we decrease the value of Δt from 2.0 s to 0.010 s, the value of the average velocity approaches the limiting value 3.84 m s^{-1} which is the value of velocity at t = 4.0 s, i.e. the value of $\frac{dx}{dt}$ at t = 4.0 s. In this manner, we can calculate velocity at each instant for motion of the car shown in Fig. 3.3. For this case, the variation of velocity with time is found to be as shown in Fig. 3.7.

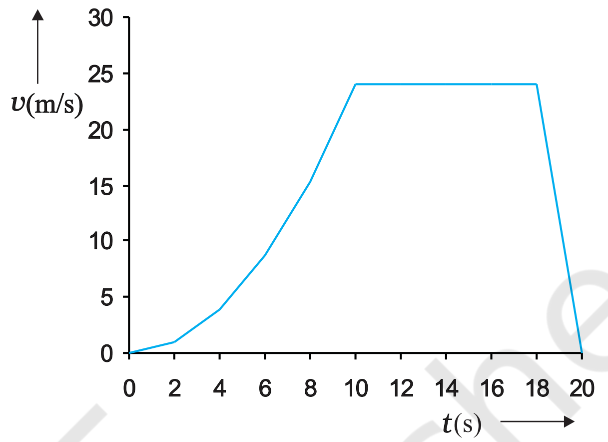


Fig. 3.7 *Velocity–time graph corresponding to motion shown in Fig. 3.3.*

The graphical method for the determination of the instantaneous velocity is always not a convenient method. For this, we must carefully plot the position–time graph and calculate the value of average velocity as Δt becomes smaller and smaller. It is easier to calculate the value of velocity at different instants if we have data of positions at different instants or exact expression for the position as a function of time. Then, we calculate $\Delta x/\Delta t$ from the data for decreasing the value of Δt and find the limiting value as we have done in Table 3.1 or use differential calculus for the given expression and calculate $\frac{dx}{dt}$ at different instants as done in the following example.

Table 3.1 **Limiting value of $\frac{\Delta x}{\Delta t}$ at t = 4 s**

Δt (s)	t_1 (s)	t_2 (s)	$x(t_1)$ (m)	$x(t_2)$ (m)	Δx (m)	$\Delta x / \Delta t$ (m s^{-1})
2.0	3.0	5.0	2.16	10.0	7.84	3.92
1.0	3.5	4.5	3.43	7.29	3.86	3.86
0.5	3.75	4.25	4.21875	6.14125	1.9225	3.845
0.1	3.95	4.05	4.93039	5.31441	0.38402	3.8402
0.01	3.995	4.005	5.100824	5.139224	0.0384	3.8400

Example 3.2 *The position of an object moving along x-axis is given by $x = a + bt^2$ where a = 8.5 m, b = 2.5 m s^{-2} and t is measured in seconds. What is its velocity at t = 0 s and t = 2.0 s. What is the average velocity between t = 2.0 s and t = 4.0 s ?*

Answer In notation of differential calculus, the velocity is

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(a + bt^2\right) = 2b\,t = 5.0\ t\ \text{m s}^{-1}$$

At t = 0 s, v = 0 m s^{-1} and at t = 2.0 s, v = 10 m s^{-1} .

$$\textit{Average velocity} = \frac{x(4.0) - x(2.0)}{4.0 - 2.0}$$

$$= \frac{a + 16b - a - 4b}{2.0} = 6.0 \times b$$

$$= 6.0 \times 2.5 = 15\ \text{m s}^{-1}$$

From Fig. 3.7, we note that during the period t =10 s to 18 s the velocity is constant. Between period t =18 s to t = 20 s, it is uniformly decreasing and during the period t = 0 s to t = 10 s, it is increasing. **Note that for uniform motion, velocity is the same as the average velocity at all instants**.

Instantaneous speed or simply speed is the magnitude of velocity. For example, a velocity of + 24.0 m s^{-1} and a velocity of – 24.0 m s^{-1} — both have an associated speed of 24.0 m s^{-1}. It should be noted that though average speed over a finite interval of time is greater or equal to the magnitude of the average velocity, instantaneous speed at an instant is equal to the magnitude of the instantaneous velocity at that instant. Why so ?

3.5 ACCELERATION

The velocity of an object, in general, changes during its course of motion. How to describe this change? Should it be described as the rate of change in velocity **with distance** or **with time** ? This was a problem even in Galileo's time. It was first thought that this change could be described by the rate of change of velocity with distance. But, through his studies of motion of freely falling objects and motion of objects on an inclined plane, Galileo concluded that the rate of change of velocity with time is a constant of motion for all objects in free fall. On the other hand, the change in velocity with distance is not constant – it decreases with the increasing distance of fall.

This led to the concept of acceleration as the rate of change of velocity with time.

The average acceleration $\bar{a}$ over a time interval is defined as the change of velocity divided by the time interval :

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t} \qquad (3.4)$$

where v_2 and v_1 are the instantaneous velocities or simply velocities at time t_2 and t_1 . It is the average change of velocity per unit time. The SI unit of acceleration is m s^{-2} .

On a plot of velocity versus time, the average acceleration is the slope of the straight line connecting the points corresponding to (v_2, t_2) and (v_1, t_1). The average acceleration for velocity-time graph shown in Fig. 3.7 for different time intervals 0 s - 10 s, 10 s – 18 s, and 18 s – 20 s are :

0 s - 10 s $\quad \bar{a} = \dfrac{(24 - 0)\,\text{m s}^{-1}}{(10 - 0)\,\text{s}} = 2.4\ \text{m s}^{-2}$

10 s - 18 s $\quad \bar{a} = \dfrac{(24 - 24)\,\text{m s}^{-1}}{(18 - 10)\,\text{s}} = 0\ \text{m s}^{-2}$

18 s - 20 s $\quad \bar{a} = \dfrac{(0 - 24)\,\text{m s}^{-1}}{(20 - 18)\,\text{s}} = -12\ \text{m s}^{-2}$

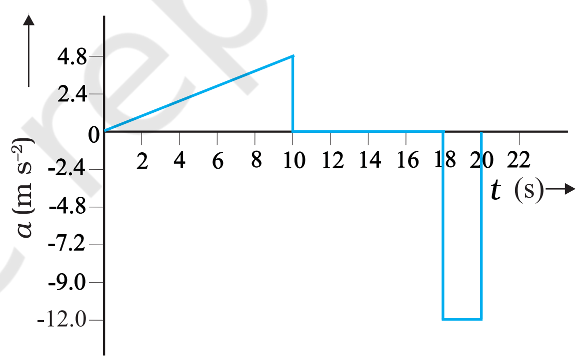


Fig. 3.8 *Acceleration as a function of time for motion represented in Fig. 3.3.*

Instantaneous acceleration is defined in the same way as the instantaneous velocity :

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \qquad (3.5)$$

The acceleration at an instant is the slope of the tangent to the v–t curve at that instant. For the v–t curve shown in Fig. 3.7, we can obtain acceleration at every instant of time. The resulting $a - t$ curve is shown in Fig. 3.8. We see that the acceleration is nonuniform over the period 0 s to 10 s. It is zero between 10 s and 18 s and is constant with value –12 m s^{-2} between 18 s and 20 s. When the acceleration is uniform, obviously, it equals the average acceleration over that period.

Since velocity is a quantity having both magnitude and direction, a change in velocity may involve either or both of these factors. Acceleration, therefore, may result from a change in speed (magnitude), a change in direction or changes in both. Like velocity, acceleration can also be positive, negative or zero. Position-time graphs for motion with positive, negative and zero acceleration are shown in Figs. 3.9 (a), (b) and (c), respectively. Note that the graph curves upward for positive acceleration; downward for negative acceleration and it is a straight line for zero acceleration. As an exercise, identify in Fig. 3.3, the regions of the curve that correspond to these three cases.

Although acceleration can vary with time, our study in this chapter will be restricted to motion with constant acceleration. In this case, the average acceleration equals the constant value of acceleration during the interval. If the velocity of an object is v_o at $t = 0$ and v at time t, we have

$$\bar{a} = \frac{v - v_0}{t - 0} \text{ or, } v = v_0 + a\,t \qquad (3.6)$$

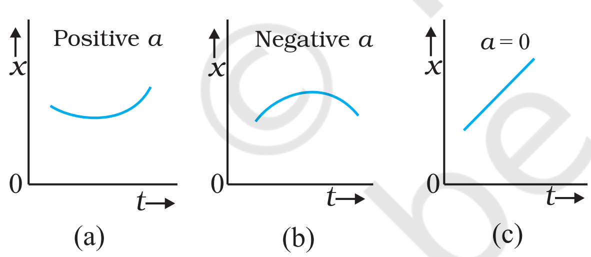


Fig. 3.9 *Position-time graph for motion with (a) positive acceleration; (b) negative acceleration, and (c) zero acceleration.*

Let us see how velocity-time graph looks like for some simple cases. Fig. 3.10 shows velocity-time graph for motion with constant acceleration for the following cases :

(a) An object is moving in a positive direction with a positive acceleration, for example the motion of the car in Fig. 3.3 between $t = 0$ s and $t = 10$ s.

(b) An object is moving in positive direction with a negative acceleration, for example, motion of the car in Fig 3.3 between $t = 18$ s and 20 s.

(c) An object is moving in negative direction with a negative acceleration, for example the motion of a car moving from O in Fig. 3.1 in negative x-direction with increasing speed.

(d) An object is moving in positive direction till time t_1, and then turns back with the same negative acceleration, for example the motion of a car from point O to point Q in Fig. 3.1 till time t_1 with decreasing speed and turning back and moving with the same negative acceleration.

An interesting feature of a velocity-time graph for any moving object is that **the area under the curve represents the displacement over a given time interval**. A general proof of this

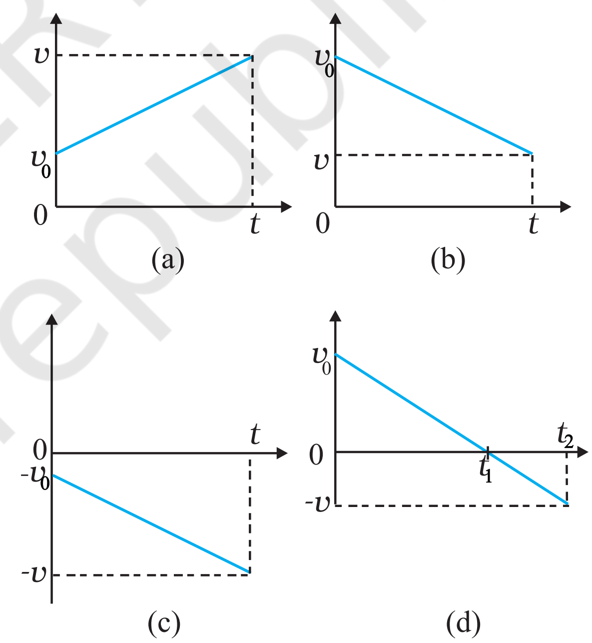


Fig. 3.10 *Velocity–time graph for motions with constant acceleration. (a) Motion in positive direction with positive acceleration, (b) Motion in positive direction with negative acceleration, (c) Motion in negative direction with negative acceleration, (d) Motion of an object with negative acceleration that changes direction at time t_1. Between times 0 to t_1, its moves in positive x - direction and between t_1 and t_2 it moves in the opposite direction.*

statement requires use of calculus. We can, however, see that it is true for the simple case of an object moving with constant velocity *u*. Its velocity-time graph is as shown in Fig. 3.11.

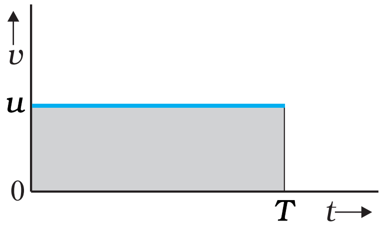


Fig. 3.11 *Area under v–t curve equals displacement of the object over a given time interval.*

The *v-t* curve is a straight line parallel to the time axis and the area under it between $t = 0$ and $t = T$ is the area of the rectangle of height *u* and base *T*. Therefore, area = $u \times T = uT$ which is the displacement in this time interval. How come in this case an area is equal to a distance? Think! Note the dimensions of quantities on the two coordinate axes, and you will arrive at the answer.

Note that the *x-t*, *v-t*, and *a-t* graphs shown in several figures in this chapter have sharp kinks at some points implying that the functions are not differentiable at these points. In any realistic situation, the functions will be differentiable at all points and the graphs will be smooth.

What this means physically is that acceleration and velocity cannot change values abruptly at an instant. Changes are always continuous.

3.6 KINEMATIC EQUATIONS FOR UNIFORMLY ACCELERATED MOTION

For uniformly accelerated motion, we can derive some simple equations that relate displacement (x), time taken (t), initial velocity (v_0), final velocity (v) and acceleration (a). Equation (3.6) already obtained gives a relation between final and initial velocities v and v_0 of an object moving with uniform acceleration a :

$$v = v_0 + at \tag{3.6}$$

This relation is graphically represented in Fig. 3.12. The area under this curve is :

Area between instants 0 and t = Area of triangle ABC + Area of rectangle OACD

$$= \frac{1}{2}(v - v_0)t + v_0 t$$

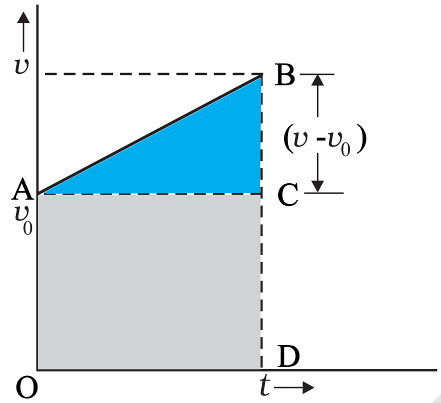


Fig. 3.12 *Area under v-t curve for an object with uniform acceleration.*

As explained in the previous section, the area under *v-t* curve represents the displacement. Therefore, the displacement x of the object is :

$$x = \frac{1}{2}(v - v_0)t + v_0 t \tag{3.7}$$

But $\quad v - v_0 = a\,t$

Therefore, $x = \frac{1}{2}a\,t^2 + v_0 t$

or, $$x = v_0 t + \frac{1}{2}at^2 \tag{3.8}$$

Equation (3.7) can also be written as

$$x = \frac{v + v_0}{2}t = \bar{v}\,t \tag{3.9a}$$

where,

$$\bar{v} = \frac{v + v_0}{2} \quad \textbf{(constant acceleration only)} \tag{3.9b}$$

Equations (3.9a) and (3.9b) mean that the object has undergone displacement x with an average velocity equal to the arithmetic average of the initial and final velocities.

From Eq. (3.6), $t = (v - v_0)/a$. Substituting this in Eq. (3.9a), we get

$$x = \bar{v}\,t = \left(\frac{v + v_0}{2}\right)\left(\frac{v - v_0}{a}\right) = \frac{v^2 - v_0^2}{2a}$$

$$v^2 = v_0^2 + 2ax \tag{3.10}$$

This equation can also be obtained by substituting the value of t from Eq. (3.6) into Eq. (3.8). Thus, we have obtained three important equations :

$$v = v_0 + at$$

$$x = v_0 t + \frac{1}{2}at^2$$

$$v^2 = v_0^2 + 2ax \qquad (3.11a)$$

connecting five quantities v_0, v, a, t and x. These are kinematic equations of rectilinear motion for constant acceleration.

The set of Eq. (3.11a) were obtained by assuming that at t = 0, the position of the particle, x is 0. We can obtain a more general equation if we take the position coordinate at t = 0 as non-zero, say x_0. Then Eqs. (3.11a) are modified (replacing x by $x - x_0$) to :

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \qquad (3.11b)$$

$$v^2 = v_0^2 + 2a(x - x_0) \qquad (3.11c)$$

Example 3.3 Obtain equations of motion for constant acceleration using method of calculus.

Answer By definition

$$a = \frac{dv}{dt}$$

$$dv = a\,dt$$

Integrating both sides

$$\int_{v_0}^{v} dv = \int_0^t a\ dt$$

$$= a\int_0^t dt \qquad (a \text{ is constant})$$

$$v - v_0 = at$$

$$v = v_0 + at$$

Further, $v = \dfrac{dx}{dt}$

$$dx = v\,dt$$

Integrating both sides

$$\int_{x_0}^{x} dx = \int_0^t v\ dt$$

$$= \int_0^t (v_0 + at)\ dt$$

$$x - x_0 = v_0\ t + \frac{1}{2}a\ \ t^2$$

$$x\ = x_0 + v_0\ t + \frac{1}{2}\ a\ t^2$$

We can write

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

or, $v\,dv = a\,dx$

Integrating both sides,

$$\int_{v_0}^{v} v\,dv\ =\ \int_{x_0}^{x} a\,dx$$

$$\frac{v^2 - v_0^2}{2}\ =\ a\,(x - x_0)$$

$$v^2 = v_0^2 + 2a\,(x - x_0)$$

The advantage of this method is that it can be used for motion with non-uniform acceleration also.

Now, we shall use these equations to some important cases.

Example 3.4 A ball is thrown vertically upwards with a velocity of 20 m s^{-1} from the top of a multistorey building. The height of the point from where the ball is thrown is 25.0 m from the ground. (a) How high will the ball rise ? and (b) how long will it be before the ball hits the ground? Take g = 10 m s^{-2}.

Answer (a) Let us take the y-axis in the vertically upward direction with zero at the ground, as shown in Fig. 3.13.

Now v_o = + 20 m s^{-1},

a = $- g$ = –10 m s^{-2},

v = 0 m s^{-1}

If the ball rises to height y from the point of launch, then using the equation

$$v^2\ =\ v_0^2 + 2\ a\ \left(y - y_0\right)$$

we get

$$0 = (20)^2 + 2(–10)(y – y_0)$$

Solving, we get, $(y – y_0)$ = 20 m.

(b) We can solve this part of the problem in two ways. **Note carefully the methods used**.

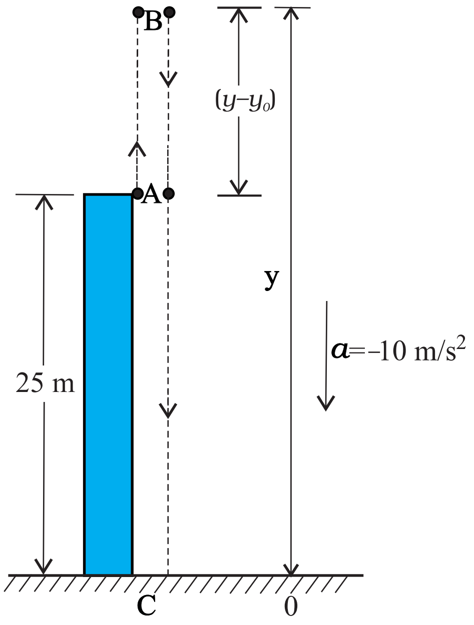


Fig. 3.13

FIRST METHOD : In the first method, we split the path in two parts : the upward motion (A to B) and the downward motion (B to C) and calculate the corresponding time taken t_1 and t_2. Since the velocity at B is zero, we have :

$$v = v_o + at$$
$$0 = 20 - 10t_1$$
Or, $$t_1 = 2 \text{ s}$$

This is the time in going from *A* to *B*. From *B*, or the point of the maximum height, the ball falls freely under the acceleration due to gravity. The ball is moving in negative *y* direction. We use equation

$$y = y_0 + v_0 t + \frac{1}{2}at^2$$

We have, $y_0 = 45$ m, $y = 0$, $v_0 = 0$, $a = -g = -10$ m s^{-2}

$$0 = 45 + (½)(-10)\, t_2^2$$

Solving, we get $t_2 = 3$ s

Therefore, the total time taken by the ball before it hits the ground = $t_1 + t_2 = 2 \text{ s} + 3 \text{ s} = 5 \text{ s}$.

SECOND METHOD : The total time taken can also be calculated by noting the coordinates of initial and final positions of the ball with respect to the origin chosen and using equation

$$y = y_0 + v_0 t + \frac{1}{2}at^2$$

Now $y_0 = 25$ m $\quad y = 0$ m
$v_o = 20$ m s^{-1}, $\quad a = -10$m s^{-2}, $\quad t = ?$

$$0 = 25 + 20\,t + (½)\,(-10)\,t^2$$

Or, $$5t^2 - 20t - 25 = 0$$

Solving this quadratic equation for *t*, we get

$$t = 5\text{s}$$

Note that the second method is better since we do not have to worry about the path of the motion as the motion is under constant acceleration.

Example 3.5 Free-fall : Discuss the motion of an object under free fall. Neglect air resistance.

Answer An object released near the surface of the Earth is accelerated downward under the influence of the force of gravity. The magnitude of acceleration due to gravity is represented by ***g***. If air resistance is neglected, the object is said to be in **free fall**. If the height through which the object falls is small compared to the earth's radius, *g* can be taken to be constant, equal to 9.8 m s^{-2}. Free fall is thus a case of motion with uniform acceleration.

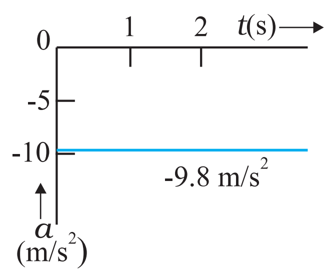
We assume that the motion is in *y*-direction, more correctly in –*y*-direction because we choose upward direction as positive. Since the acceleration due to gravity is always downward, it is in the negative direction and we have

$$a = -g = -9.8 \text{ m s}^{-2}$$

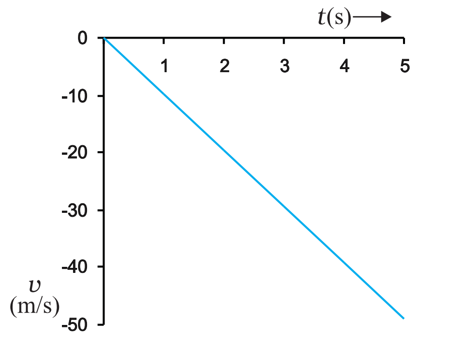
The object is released from rest at $y = 0$. Therefore, $v_0 = 0$ and the equations of motion become:

$$v = 0 - g\,t \quad = -9.8\,t \quad \text{m s}^{-1}$$
$$y = 0 - ½\,g\,t^2 \quad = -4.9\,t^2 \quad \text{m}$$
$$v^2 = 0 - 2\,g\,y \quad = -19.6\,y \quad \text{m}^2\,\text{s}^{-2}$$

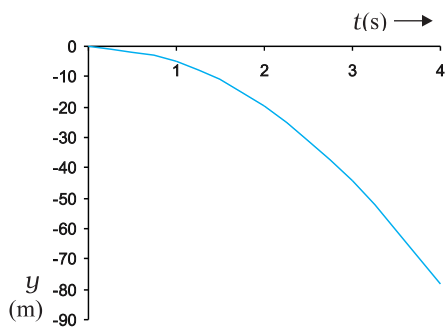
These equations give the velocity and the distance travelled as a function of time and also the variation of velocity with distance. The variation of acceleration, velocity, and distance, with time have been plotted in Fig. 3.14(a), (b) and (c).



(a)



(b)



(c)

Fig. 3.14 *Motion of an object under free fall.*
(a) Variation of acceleration with time.
(b) Variation of velocity with time.
(c) Variation of distance with time

***Example 3.6* Galileo's law of odd numbers** : *"The distances traversed, during equal intervals of time, by a body falling from rest, stand to one another in the same ratio as the odd numbers beginning with unity [namely, 1: 3: 5: 7......]." Prove it.*

Answer Let us divide the time interval of motion of an object under free fall into many equal intervals τ and find out the distances traversed during successive intervals of time. Since initial velocity is zero, we have

$$y = -\frac{1}{2} g t^2$$

Using this equation, we can calculate the position of the object after different time intervals, 0, τ, 2τ, 3τ... which are given in second column of Table 3.2. If we take $(-1/2)\, g\tau^2$ as y_0 — the position coordinate after first time interval τ, then third column gives the positions in the unit of y_o. The fourth column gives the distances traversed in successive τs. We find that the distances are in the simple ratio 1: 3: 5: 7: 9: 11... as shown in the last column. This law was established by Galileo Galilei (1564-1642) who was the first to make quantitative studies of free fall.

***Example 3.7* Stopping distance of vehicles :** When brakes are applied to a moving vehicle, the distance it travels before stopping is called stopping distance. It is an important factor for road safety and depends on the initial velocity (v_0) and the braking capacity, or deceleration, $-a$ that is caused by the braking. Derive an expression for stopping distance of a vehicle in terms of v_o and a.

Answer Let the distance travelled by the vehicle before it stops be d_s. Then, using equation of motion $v^2 = v_o^2 + 2\,ax$, and noting that $v = 0$, we have the stopping distance

$$d_s = \frac{-v_0^2}{2a}$$

Thus, the stopping distance is proportional to the square of the initial velocity. Doubling the

Table 3.2

t	y	y in terms of y_o [=(- ½) g τ^2]	Distance traversed in successive intervals	Ratio of distances traversed
0	0	0		
τ	$-(1/2)\, g\, \tau^2$	y_o	y_o	1
2 τ	$-4(1/2)\, g\, \tau^2$	4 y_o	3 y_o	3
3 τ	$-9(1/2)\, g\, \tau^2$	9 y_o	5 y_o	5
4 τ	$-16(1/2)\, g\, \tau^2$	16 y_o	7 y_o	7
5 τ	$-25(1/2)\, g\, \tau^2$	25 y_o	9 y_o	9
6 τ	$-36(1/2)\, g\, \tau^2$	36 y_o	11 y_o	11

initial velocity increases the stopping distance by a factor of 4 (for the same deceleration).

For the car of a particular make, the braking distance was found to be 10 m, 20 m, 34 m and 50 m corresponding to velocities of 11, 15, 20 and 25 m/s which are nearly consistent with the above formula.

Stopping distance is an important factor considered in setting speed limits, for example, in school zones. ◀

▶ ***Example 3.8* Reaction time :** When a situation demands our immediate action, it takes some time before we really respond. Reaction time is the time a person takes to observe, think and act. For example, if a person is driving and suddenly a boy appears on the road, then the time elapsed before he slams the brakes of the car is the reaction time. Reaction time depends on complexity of the situation and on an individual.

You can measure your reaction time by a simple experiment. Take a ruler and ask your friend to drop it vertically through the gap between your thumb and forefinger (Fig. 3.15). After you catch it, find the distance d travelled by the ruler. In a particular case, d was found to be 21.0 cm. Estimate reaction time.

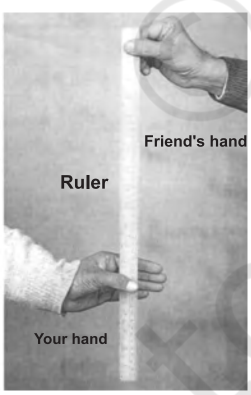


Fig. 3.15 *Measuring the reaction time.*

Answer The ruler drops under free fall. Therefore, $v_o = 0$, and $a = -g = -9.8$ m s^{-2}. The distance travelled d and the reaction time t_r are related by

$$d = -\frac{1}{2} g t_r^2$$

Or, $$t_r = \sqrt{\frac{2d}{g}} \text{ s}$$

Given $d = 21.0$ cm and $g = 9.8$ m s^{-2} the reaction time is

$$t_r = \sqrt{\frac{2 \times 0.21}{9.8}} \text{ s} \cong 0.2 \text{ s.}$$ ◀

3.7 RELATIVE VELOCITY

You must be familiar with the experience of travelling in a train and being overtaken by another train moving in the same direction as you are. While that train must be travelling faster than you to be able to pass you, it does seem slower to you than it would be to someone standing on the ground and watching both the trains. In case both the trains have the same velocity with respect to the ground, then to you the other train would seem to be not moving at all. To understand such observations, we now introduce the concept of relative velocity.

Consider two objects A and B moving uniformly with average velocities v_A and v_B in one dimension, say along x-axis. (Unless otherwise specified, the velocities mentioned in this chapter are measured with reference to the ground). If $x_A(0)$ and $x_B(0)$ are positions of objects A and B, respectively at time $t = 0$, their positions $x_A(t)$ and $x_B(t)$ at time t are given by:

$$x_A(t) = x_A(0) + v_A t \quad (3.12a)$$
$$x_B(t) = x_B(0) + v_B t \quad (3.12b)$$

Then, the displacement from object A to object B is given by

$$x_{BA}(t) = x_B(t) - x_A(t)$$
$$= [\, x_B(0) - x_A(0)\,] + (v_B - v_A)\, t. \quad (3.13)$$

Equation (3.13) is easily interpreted. It tells us that as seen from object A, object B has a velocity $v_B - v_A$ because the displacement from A to B changes steadily by the amount $v_B - v_A$ in each unit of time. We say that the velocity of object B *relative to object* A is $v_B - v_A$:

$$v_{BA} = v_B - v_A \quad (3.14a)$$

Similarly, velocity of object A *relative to object* B is:

$$v_{AB} = v_A - v_B \quad (3.14b)$$