CHAPTER 12

SIMPLE HARMONIC MOTION

12.1 SIMPLE HARMONIC MOTION

When a body repeats its motion after regular time intervals we say that it is in *harmonic motion* or *periodic motion*. The time interval after which the motion is repeated is called the *time period*. If a body moves to and fro on the same path, it is said to perform *oscillations*. *Simple harmonic motion* (SHM) is a special type of oscillation in which the particle oscillates on a straight line, the acceleration of the particle is always directed towards a fixed point on the line and its magnitude is proportional to the displacement of the particle from this point. This fixed point is called the *centre of oscillation*. Taking this point as the origin and the line of motion as the X-axis, we can write the defining equation of a simple harmonic motion as

$$a = -\omega^2 x \qquad \dots (12.1)$$

where ω^2 is a positive constant. If x is positive, a is negative and if x is negative, a is positive. This means that the acceleration is always directed towards the centre of oscillation.

If we are looking at the motion from an inertial frame,

$$a = F/m.$$

The defining equation (12.1) may thus be written as

$$F/m = -\omega^2 x$$

or, $$F = -m\omega^2 x$$

or, $$F = -kx. \qquad \dots (12.2)$$

We can use equation (12.2) as the definition of SHM. A particle moving on a straight line executes simple harmonic motion if the resultant force acting on it is directed towards a fixed point on the line and is proportional to the displacement of the particle from this fixed point. The constant $k = m\omega^2$ is called the *force constant* or *spring constant*. The resultant force on the particle is zero when it is at the centre of oscillation. The centre of oscillation is, therefore, the equilibrium position. A force which takes the particle back towards the equilibrium position is called a *restoring force*. Equation (12.2) represents a restoring force which is linear. Figure (12.1) shows the linear restoring force graphically.

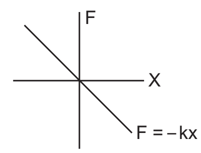

Figure 12.1

Example 12.1

The resultant force acting on a particle executing simple harmonic motion is 4 N *when it is* 5 cm *away from the centre of oscillation. Find the spring constant.*

Solution : The simple harmonic motion is defined as

$$F = -k\,x.$$

The spring constant is $k = \left|\dfrac{F}{x}\right|$

$$= \frac{4\ \text{N}}{5\ \text{cm}} = \frac{4\ \text{N}}{5\times 10^{-2}\ \text{m}} = 80\ \text{N m}^{-1}.$$

12.2 QUALITATIVE NATURE OF SIMPLE HARMONIC MOTION

Let us consider a small block of mass m placed on a smooth horizontal surface and attached to a fixed wall through a spring as shown in figure (12.2). Let the spring constant of the spring be k.

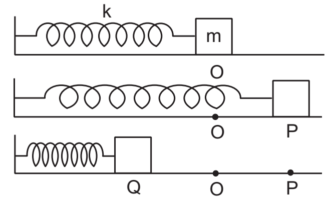

Figure 12.2

The block is at a position O when the spring is at its natural length. Suppose the block is taken to a point P stretching the spring by the distance $OP = A$ and is released from there.

At any point on its path the displacement x of the particle is equal to the extension of the spring from its natural length. The resultant force on the particle is given by $F = -kx$ and hence by definition the motion of the block is simple harmonic.

When the block is released from P, the force acts towards the centre O. The block is accelerated in that direction. The force continues to act towards O until the block reaches O. The speed thus increases all the time from P to O. When the block reaches O, its speed is maximum and it is going towards left. As it moves towards left from O, the spring becomes compressed. The spring pushes the block towards right and hence its speed decreases. The block moves to a point Q when its speed becomes zero. The potential energy of the system (block + spring), when the block is at P, is $\frac{1}{2}k\,(OP)^2$ and when the block is at Q it is $\frac{1}{2}k\,(OQ)^2$. Since the block is at rest at P as well as at Q, the kinetic energy is zero at both these positions. As we have assumed frictionless surface, principle of conservation of energy gives

$$\frac{1}{2}k\,(OP)^2 = \frac{1}{2}k\,(OQ)^2$$

or, $$OP = OQ.$$

The spring is now compressed and hence it pushes the block towards right. The block starts moving towards right, its speed increases upto O and then decreases to zero when it reaches P. Thus the particle oscillates between P and Q. As $OP = OQ$, it moves through equal distances on both sides of the centre of oscillation. The maximum displacement on either side from the centre of oscillation is called the *amplitude*.

Example 12.2

A particle of mass 0·50 kg *executes a simple harmonic motion under a force* $F = -(50\ \mathrm{N\ m^{-1}})x$. *If it crosses the centre of oscillation with a speed of* 10 m s^{-1}, *find the amplitude of the motion.*

Solution **:** The kinetic energy of the particle when it is at the centre of oscillation is $E = \frac{1}{2}m v^2$

$$= \frac{1}{2}(0{\cdot}50\ \mathrm{kg})\,(10\ \mathrm{m\ s^{-1}})^2$$

$$= 25\ \mathrm{J}.$$

The potential energy is zero here. At the maximum displacement $x = A$, the speed is zero and hence the kinetic energy is zero. The potential energy here is $\frac{1}{2}k A^2$. As there is no loss of energy,

$$\frac{1}{2}k A^2 = 25\ \mathrm{J}. \qquad \text{... (i)}$$

The force on the particle is given by

$$F = -(50\ \mathrm{N\ m^{-1}})x.$$

Thus, the spring constant is $k = 50\ \mathrm{N\ m^{-1}}$.

Equation (i) gives

$$\frac{1}{2}(50\ \mathrm{N\ m^{-1}})\,A^2 = 25\ \mathrm{J}$$

or, $$A = 1\ \mathrm{m}.$$

12.3 EQUATION OF MOTION OF A SIMPLE HARMONIC MOTION

Consider a particle of mass m moving along the X-axis. Suppose, a force $F = -kx$ acts on the particle where k is a positive constant and x is the displacement of the particle from the assumed origin. The particle then executes a simple harmonic motion with the centre of oscillation at the origin. We shall calculate the displacement x and the velocity v as a function of time.

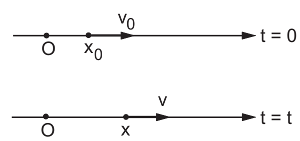

Figure 12.3

Suppose the position of the particle at $t = 0$ is x_0 and its velocity is v_0. Thus,

at $t = 0, x = x_0$ and $v = v_0$.

The acceleration of the particle at any instant is

$$a = \frac{F}{m} = -\frac{k}{m}x = -\omega^2 x$$

where $\omega = \sqrt{km^{-1}}$.

Thus, $$\frac{dv}{dt} = -\omega^2 x \qquad \text{... (12.3)}$$

or, $$\frac{dv}{dx}\frac{dx}{dt} = -\omega^2 x$$

or, $$v\frac{dv}{dx} = -\omega^2 x$$

or, $$v\,dv = -\omega^2 x\,dx.$$

The velocity of the particle is v_0 when the particle is at $x = x_0$. It becomes v when the displacement becomes x. We can integrate the above equation and write

$$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} -\omega^2 x\,dx$$

or, $$\left[\frac{v^2}{2}\right]_{v_0}^{v} = -\omega^2\left[\frac{x^2}{2}\right]_{x_0}^{x}$$

or, $$v^2 - v_0^2 = -\omega^2(x^2 - x_0^2)$$

or, $$v^2 = (v_0^2 + \omega^2 x_0^2 - \omega^2 x^2)$$

or, $$v = \sqrt{(v_0^2 + \omega^2 x_0^2) - \omega^2 x^2}$$

or, $$v = \omega\sqrt{\left(\frac{v_0^2}{\omega^2} + x_0^2\right) - x^2}\,.$$

Writing $$\left(\frac{v_0}{\omega}\right)^2 + x_0^2 = A^2 \qquad \ldots\ (12.4)$$

the above equation becomes

$$v = \omega\sqrt{A^2 - x^2}. \qquad \ldots\ (12.5)$$

We can write this equation as

$$\frac{dx}{dt} = \omega\sqrt{A^2 - x^2}$$

or, $$\frac{dx}{\sqrt{A^2 - x^2}} = \omega\, dt.$$

At time $t = 0$ the displacement is $x = x_0$ and at time t the displacement becomes x. The above equation can be integrated as

$$\int_{x_0}^{x} \frac{dx}{\sqrt{A^2 - x^2}} = \int_0^t \omega\, dt$$

or, $$\left[\sin^{-1}\frac{x}{A}\right]_{x_0}^{x} = [\omega t]_0^t$$

or, $$\sin^{-1}\frac{x}{A} - \sin^{-1}\frac{x_0}{A} = \omega t.$$

Writing $\sin^{-1}\frac{x_0}{A} = \delta$, this becomes

$$\sin^{-1}\frac{x}{A} = \omega t + \delta$$

or, $$x = A\sin(\omega t + \delta). \qquad \ldots\ (12.6)$$

The velocity at time t is

$$v = \frac{dx}{dt} = A\,\omega\cos(\omega t + \delta). \qquad \ldots\ (12.7)$$

12.4 TERMS ASSOCIATED WITH SIMPLE HARMONIC MOTION

(a) Amplitude

Equation (12.6) gives the displacement of a particle in simple harmonic motion. As $\sin(\omega t + \delta)$ can take values between -1 and $+1$, the displacement x can take values between $-A$ and $+A$. This gives the physical significance of the constant A. It is the maximum displacement of the particle from the centre of oscillation, i.e, the amplitude of oscillation.

(b) Time Period

A particle in simple harmonic motion repeats its motion after a regular time interval. Suppose the particle is at a position x and its velocity is v at a certain time t. After some time the position of the particle will again be x and its velocity will again be v in the same direction. This part of the motion is called *one complete oscillation* and the time taken in one complete oscillation is called the *time period T*. Thus, in figure (12.4) Q to P and then back to Q is a complete oscillation, R to P to Q to R is a complete oscillation, O to P to Q to O is a complete oscillation, etc. Both the position and the velocity (magnitude as well as direction) repeat after each complete oscillation.

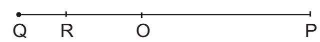

Figure 12.4

We have,

$$x = A\sin(\omega t + \delta).$$

If T be the time period, x should have same value at t and $t + T$.

Thus, $$\sin(\omega t + \delta) = \sin[\omega(t + T) + \delta].$$

Now the velocity is (equation 12.7)

$$v = A\,\omega\cos(\omega t + \delta).$$

As the velocity also repeats its value after a time period, $\cos(\omega t + \delta) = \cos[\omega(t + T) + \delta]$.

Both $\sin(\omega t + \delta)$ and $\cos(\omega t + \delta)$ will repeat their values if the angle $(\omega t + \delta)$ increases by 2π or its multiple. As T is the smallest time for repetition,

$$\omega(t + T) + \delta = (\omega t + \delta) + 2\pi$$

or, $$\omega T = 2\pi$$

or, $$T = \frac{2\pi}{\omega}\cdot$$

Remembering that $\omega = \sqrt{km^{-1}}$, we can write for the time period,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad \ldots\ (12.8)$$

where k is the force constant and m is the mass of the particle.

Example 12.3

A particle of mass 200 g *executes a simple harmonic motion. The restoring force is provided by a spring of spring constant* 80 N m^{-1}. *Find the time period.*

Solution : The time period is

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$= 2\pi \sqrt{\frac{200 \times 10^{-3}\ \mathrm{kg}}{80\ \mathrm{N\ m^{-1}}}}$$

$$= 2\pi \times 0{\cdot}05\ \mathrm{s} = 0{\cdot}31\ \mathrm{s}.$$

(c) Frequency and Angular Frequency

The reciprocal of time period is called the *frequency*. Physically, the frequency represents the number of oscillations per unit time. It is measured in cycles per second also known as *hertz* and written in symbols as Hz. Equation (12.8) shows that the frequency is

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \qquad \dots (12.9)$$

$$= \frac{1}{2\pi}\sqrt{\frac{k}{m}}\,. \qquad \dots (12.10)$$

The constant ω is called the *angular frequency*.

(d) Phase

The quantity $\phi = \omega t + \delta$ is called the phase. It determines the status of the particle in simple harmonic motion. If the phase is zero at a certain instant, $x = A \sin(\omega t + \delta) = 0$ and $v = A\,\omega \cos(\omega t + \delta) = A\,\omega$. This means that the particle is crossing the mean position and is going towards the positive direction. If the phase is $\pi/2$, we get $x = A$, $v = 0$ so that the particle is at the positive extreme position. Figure (12.5) shows the status of the particle at different phases.

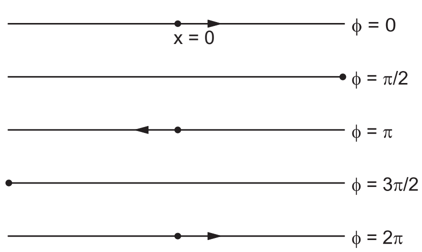

Figure 12.5

We see that as time increases the phase increases. An increase of 2π brings the particle to the same status in the motion. Thus, a phase $\omega t + \delta$ is equivalent to a phase $\omega t + \delta + 2\pi$. Similarly, a phase change of 4π, 6π, 8π, …, etc., are equivalent to no phase change.

Figure (12.6) shows graphically the variation of position and velocity as a function of the phase.

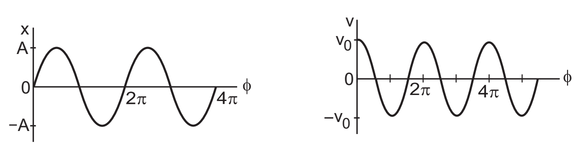

Figure 12.6

(e) Phase constant

The constant δ appearing in equation (12.6) is called the *phase constant*. This constant depends on the choice of the instant $t = 0$. To describe the motion quantitatively, a particular instant should be called $t = 0$ and measurement of time should be made from this instant. This instant may be chosen according to the convenience of the problem. Suppose we choose $t = 0$ at an instant when the particle is passing through its mean position and is going towards the positive direction. The phase $\omega t + \delta$ should then be zero. As $t = 0$ this means δ will be zero. The equation for displacement can then be written as

$$x = A \sin\omega t.$$

If we choose $t = 0$ at an instant when the particle is at its positive extreme position, the phase is $\pi/2$ at this instant. Thus $\omega t + \delta = \pi/2$ and hence $\delta = \pi/2$. The equation for the displacement is $x = A \sin(\omega t + \pi/2)$

or, $\quad x = A \cos\omega t.$

Any instant can be chosen as $t = 0$ and hence the phase constant can be chosen arbitrarily. Quite often we shall choose $\delta = 0$ and write the equation for displacement as $x = A \sin\omega t$. Sometimes we may have to consider two or more simple harmonic motions together. The phase constant of any one can be chosen as $\delta = 0$. The phase constants of the rest will be determined by the actual situation. The general equation for displacement may be written as

$$x = A \sin(\omega t + \delta)$$

$$= A \sin\left(\omega t + \frac{\pi}{2} + \delta'\right)$$

$$= A \cos(\omega t + \delta')$$

where δ' is another arbitrary constant. The sine form and the cosine form are, therefore, equivalent. The value of phase constant, however, depends on the form chosen.

Example 12.4

A particle executes simple harmonic motion of amplitude A along the X-axis. At $t = 0$, the position of the particle is $x = A/2$ and it moves along the positive x-direction. Find the phase constant δ if the equation is written as $x = A \sin(\omega t + \delta)$.

Solution : We have $x = A \sin(\omega t + \delta)$. At $t = 0$, $x = A/2$.

Thus, $\quad A/2 = A \sin\delta$

or, $\quad \sin\delta = 1/2$

or, $\quad \delta = \pi/6 \ \text{ or } \ 5\pi/6.$

The velocity is $v = \dfrac{dx}{dt} = A\,\omega \cos(\omega t + \delta)$.

At $\quad t = 0,\ v = A\,\omega \cos\delta.$

Now, $\quad \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$ and $\cos\dfrac{5\pi}{6} = -\dfrac{\sqrt{3}}{2}$.

As v is positive at $t = 0$, δ must be equal to $\pi/6$.

12.5 SIMPLE HARMONIC MOTION AS A PROJECTION OF CIRCULAR MOTION

Consider a particle P moving on a circle of radius A with a constant angular speed ω (figure 12.7). Let us take the centre of the circle as the origin and two perpendicular diameters as the X and Y-axes. Suppose the particle P is on the X-axis at $t = 0$. The radius OP will make an angle $\theta = \omega t$ with the X-axis at time t. Drop perpendicular PQ on X-axis and PR on Y-axis. The x and y-coordinates of the particle at time t are

$$x = OQ = OP \cos \omega t$$

or, $$x = A \cos \omega t \qquad \dots \quad (12.11)$$

and $$y = OR = OP \sin \omega t$$

or, $$y = A \sin \omega t. \qquad \dots \quad (12.12)$$

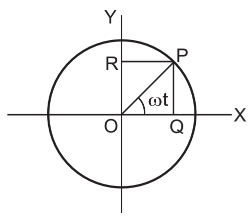

Figure 12.7

Equation (12.11) shows that the foot of perpendicular Q executes a simple harmonic motion on the X-axis. The amplitude is A and the angular frequency is ω. Similarly, equation (12.12) shows that the foot of perpendicular R executes a simple harmonic motion on the Y-axis. The amplitude is A and the angular frequency is ω. The phases of the two simple harmonic motions differ by $\pi/2$ [remember $\cos\omega t = \sin(\omega t + \pi/2)$].

Thus, the projection of a uniform circular motion on a diameter of the circle is a simple harmonic motion.

12.6 ENERGY CONSERVATION IN SIMPLE HARMONIC MOTION

Simple harmonic motion is defined by the equation

$$F = -kx.$$

The work done by the force F during a displacement from x to $x + dx$ is

$$dW = F\,dx$$
$$= -kx\,dx.$$

The work done in a displacement from $x = 0$ to x is

$$W = \int_0^x (-kx)dx = -\frac{1}{2}kx^2.$$

Let $U(x)$ be the potential energy of the system when the displacement is x. As the change in potential energy corresponding to a force is negative of the work done by this force,

$$U(x) - U(0) = -W = \frac{1}{2}kx^2.$$

Let us choose the potential energy to be zero when the particle is at the centre of oscillation $x = 0$.

Then $$U(0) = 0 \text{ and } U(x) = \frac{1}{2}kx^2.$$

This expression for potential energy is same as that for a spring and has been used so far in this chapter.

As $$\omega = \sqrt{\frac{k}{m}}, \; k = m\omega^2$$

we can write $$U(x) = \frac{1}{2}m\omega^2 x^2. \qquad \dots \quad (12.13)$$

The displacement and the velocity of a particle executing a simple harmonic motion are given by

$$x = A\sin(\omega t + \delta)$$

and $$v = A\omega\cos(\omega t + \delta).$$

The potential energy at time t is, therefore,

$$U = \frac{1}{2}m\omega^2 x^2$$
$$= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \delta),$$

and the kinetic energy at time t is

$$K = \frac{1}{2}mv^2$$
$$= \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \delta).$$

The total mechanical energy at time t is

$$E = U + K$$
$$= \frac{1}{2}m\omega^2 A^2[\sin^2(\omega t + \delta) + (\cos^2(\omega t + \delta)]$$
$$= \frac{1}{2}m\omega^2 A^2. \qquad \dots \quad (12.14)$$

We see that the total mechanical energy at time t is independent of t. Thus, the mechanical energy remains constant as expected.

As an example, consider a small block of mass m placed on a smooth horizontal surface and attached to a fixed wall through a spring of spring constant k (figure 12.8).

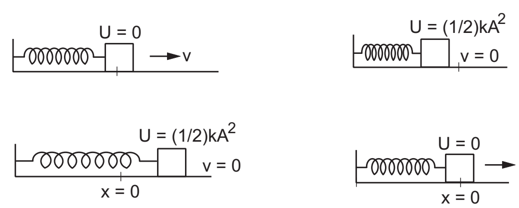

Figure 12.8

When displaced from the mean position (where the spring has its natural length), the block executes a simple harmonic motion. The spring is the agency exerting a force $F = -kx$ on the block. The potential energy of the system is the elastic potential energy stored in the spring.

At the mean position $x = 0$, the potential energy is zero. The kinetic energy is $\frac{1}{2} m v_0^2 = \frac{1}{2} m \omega^2 A^2$. All the mechanical energy is in the form of kinetic energy here. As the particle is displaced away from the mean position, the kinetic energy decreases and the potential energy increases. At the extreme positions $x = \pm A$, the speed v is zero and the kinetic energy decreases to zero. The potential energy is increased to its maximum value $\frac{1}{2} kA^2 = \frac{1}{2} m \omega^2 A^2$. All the mechanical energy is in the form of potential energy here.

Example 12.5

A particle of mass 40 g *executes a simple harmonic motion of amplitude* 2·0 cm. *If the time period is* 0·20 s, *find the total mechanical energy of the system.*

Solution : The total mechanical energy of the system is

$$E = \frac{1}{2} m \omega^2 A^2$$

$$= \frac{1}{2} m \left(\frac{2\pi}{T}\right)^2 A^2 = \frac{2\pi^2 m A^2}{T^2}$$

$$= \frac{2\pi^2 (40 \times 10^{-3}\ \text{kg})\ (2{\cdot}0 \times 10^{-2}\ \text{m})^2}{(0{\cdot}20\ \text{s})^2}$$

$$= 7{\cdot}9 \times 10^{-3}\ \text{J}.$$

12.7 ANGULAR SIMPLE HARMONIC MOTION

A body free to rotate about a given axis can make angular oscillations. For example, a hanging umbrella makes angular oscillations when it is slightly pushed aside and released. The angular oscillations are called angular simple harmonic motion if

(a) there is a position of the body where the resultant torque on the body is zero, this position is the mean position $\theta = 0$,

(b) when the body is displaced through an angle from the mean position, a resultant torque acts which is proportional to the angle displaced, and

(c) this torque has a sense (clockwise or anticlockwise) so as to bring the body towards the mean position.

If the angular displacement of the body at an instant is θ, the resultant torque acting on the body in angular simple harmonic motion should be

$$\Gamma = -k\,\theta.$$

If the moment of inertia is I, the angular acceleration is

$$\alpha = \frac{\Gamma}{I} = -\frac{k}{I}\,\theta$$

or,
$$\frac{d^2\theta}{dt^2} = -\omega^2\,\theta \qquad \ldots \quad (12.15)$$

where $\omega = \sqrt{kI^{-1}}$.

Equation (12.15) is identical to equation (12.3) except for the symbols. The linear displacement x in (12.3) is replaced here by the angular displacement θ. Thus, equation (12.15) may be integrated in the similar manner and we shall get an equation similar to (12.6), i.e.,

$$\theta = \theta_0 \sin(\omega t + \delta) \qquad \ldots \quad (12.16)$$

where θ_0 is the maximum angular displacement on either side. The angular velocity at time t is given by,

$$\Omega = \frac{d\theta}{dt} = \theta_0\,\omega \cos(\omega t + \delta). \qquad \ldots \quad (12.17)$$

The time period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{k}} \qquad \ldots \quad (12.18)$$

and the frequency of oscillation is

$$\nu = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{I}}\,. \qquad \ldots \quad (12.19)$$

The quantity $\omega = \sqrt{kI^{-1}}$ is the angular frequency.

Example 12.6

A body makes angular simple harmonic motion of amplitude $\pi/10$ rad *and time period* 0·05 s. *If the body is at a displacement* $\theta = \pi/10$ rad *at* $t = 0$, *write the equation giving the angular displacement as a function of time.*

Solution : Let the required equation be

$$\theta = \theta_0 \sin(\omega t + \delta).$$

Here $\theta_0 =$ amplitude $= \frac{\pi}{10}$ rad

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0{\cdot}05\ \text{s}} = 40\,\pi\ \text{s}^{-1}$$

so that
$$\theta = \left(\frac{\pi}{10}\ \text{rad}\right) \sin\left[\left(40\,\pi\ \text{s}^{-1}\right) t + \delta\right]. \qquad \ldots \quad \text{(i)}$$

At $t = 0$, $\theta = \pi/10$ rad. Putting in (i),

$$\frac{\pi}{10} = \left(\frac{\pi}{10}\right) \sin\delta$$

or, $\sin\delta = 1$

or, $\delta = \pi/2$.

Thus by (i),

$$\theta = \left(\frac{\pi}{10}\ \text{rad}\right) \sin\left[(40\,\pi\ \text{s}^{-1}) t + \frac{\pi}{2}\right]$$

$$= \left(\frac{\pi}{10}\ \text{rad}\right) \cos[(40\,\pi\ \text{s}^{-1})\ t\ ].$$

Energy

The potential energy is

$$U = \frac{1}{2} k\,\theta^2 = \frac{1}{2} I\,\omega^2\,\theta^2$$

and the kinetic energy is

$$K = \frac{1}{2} I\,\Omega^2.$$

The total energy is

$$E = U + K$$

$$= \frac{1}{2} I\,\omega^2\,\theta^2 + \frac{1}{2} I\,\Omega^2.$$

Using $\theta = \theta_0 \sin(\omega t + \delta)$

$$E = \frac{1}{2} I\,\omega^2\,\theta_0^2 \sin^2(\omega t + \delta) + \frac{1}{2} I\,\theta_0^2\,\omega^2 \cos^2(\omega t + \delta)$$

$$= \frac{1}{2} I\,\omega^2 \theta_0^2. \qquad \ldots (12.20)$$

12.8 SIMPLE PENDULUM

A simple pendulum consists of a heavy particle suspended from a fixed support through a light inextensible string. Simple pendulum is an idealised model. In practice, one takes a small metallic sphere and suspends it through a string.

Figure (12.9) shows a simple pendulum in which a particle of mass m is suspended from the fixed support O through a light string of length l. The system can stay in equilibrium if the string is vertical. This is the mean or equilibrium position. If the particle is pulled aside and released, it oscillates in a circular arc with the centre at the point of suspension O.

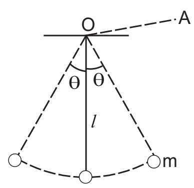

Figure 12.9

The position of the particle at any time can be described by the angle θ between the string and the vertical. The mean position or the equilibrium position corresponds to $\theta = 0$. The particle makes pure rotation about the horizontal line OA (figure 12.9) which is perpendicular to the plane of motion.

Let us see whether the motion of the particle is simple harmonic or not and find out its time period of oscillation.

Let the particle be at P at a time t when the string OP makes an angle θ with the vertical (figure 12.10). Let OQ be the horizontal line in the plane of motion. Let PQ be the perpendicular to OQ.

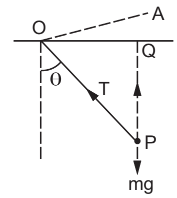

Figure 12.10

Forces acting on the particle are (a) the weight mg and (b) the tension T.

The torque of T about OA is zero as it intersects OA. The magnitude of the torque of mg about OA is

$$|\Gamma| = (mg)\,(OQ)$$

$$= mg\,(OP)\sin\theta$$

$$= mgl\sin\theta.$$

Also, the torque tries to bring the particle back towards $\theta = 0$. Thus, we can write

$$\Gamma = -\,mgl\sin\theta. \qquad \ldots (12.21)$$

We see that the resultant torque is not proportional to the angular displacement and hence the motion is not angular simple harmonic. However, if the angular displacement is small, $\sin\theta$ is approximately equal to θ (expressed in radians) and equation (12.21) may be written as

$$\Gamma = -\,mgl\,\theta. \qquad \ldots (12.22)$$

Thus, if the amplitude of oscillation is small, the motion of the particle is approximately angular simple harmonic. The moment of inertia of the particle about the axis of rotation OA is

$$I = m(OP)^2 = ml^2.$$

The angular acceleration is

$$\alpha = \frac{\Gamma}{I} = -\frac{mgl\,\theta}{ml^2} = -\frac{g}{l}\,\theta$$

or, $\alpha = -\,\omega^2\,\theta$

where $\omega = \sqrt{gl^{-1}}$.

This is the equation of an angular simple harmonic motion. The constant $\omega = \sqrt{gl^{-1}}$ represents the angular frequency. The time period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{l/g}. \qquad \ldots (12.23)$$

Example 12.7

Calculate the time period of a simple pendulum of length one meter. The acceleration due to gravity at the place is $\pi^2\ \mathrm{m\,s^{-2}}$.

Solution **:** The time period is

$$T = 2\pi \sqrt{lg^{-1}}$$

$$= 2\pi \sqrt{\frac{1{\cdot}00 \text{ m}}{\pi^2 \text{ m s}^{-2}}} = 2{\cdot}0 \text{ s}.$$

Simple Pendulum as a Linear Simple Harmonic Oscillator

If the amplitude of oscillation is small, the path of the particle is approximately a straight line and the motion can be described as a linear simple harmonic motion. We rederive expression (12.23) for the time period using this approach.

Consider the situation shown in figure (12.11).

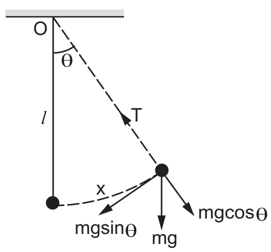

Figure 12.11

Suppose the string makes an angle θ with the vertical at time t. The distance of the particle from the equilibrium position along the arc is $x = l\theta$. The speed of the particle at time t is

$$v = \frac{dx}{dt}$$

and the tangential acceleration is

$$a_t = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \quad \text{... (i)}$$

Forces acting on the particle are (a) the weight mg and (b) the tension T. The component of mg along the tangent to the path is $-mg\sin\theta$ and that of T is zero. Thus, the total tangential force on the particle is $-mg\sin\theta$. Using (i) we get

$$-mg\sin\theta = m\frac{d^2x}{dt^2}$$

or, $$\frac{d^2x}{dt^2} = -g\sin\theta. \quad \text{... (ii)}$$

If the amplitude of oscillation is small, $\sin\theta \approx \theta = xl^{-1}$. Equation (ii) above thus becomes (for small oscillations)

$$\frac{d^2x}{dt^2} = -\frac{g}{l}x$$

or, $$\frac{d^2x}{dt^2} = -\omega^2 x$$

where $$\omega = \sqrt{gl^{-1}}.$$

This equation represents a simple harmonic motion of the particle along the arc of the circle in which it moves. The angular frequency is $\omega = \sqrt{gl^{-1}}$ and the time period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{lg^{-1}}$$

which is same as in equation (12.23).

Determination of *g* in Laboratory

A simple pendulum provides an easy method to measure the value of '*g*' in a laboratory. A small spherical ball with a hook is suspended from a clamp through a light thread as shown in figure (12.12).

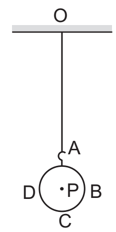

Figure 12.12

The lengths AC and BD are measured with slide callipers. The length OA of the thread is measured with a meter scale. The effective length is

$$OP = OA + AP = OA + AC - \frac{BD}{2}.$$

The bob is slightly pulled aside and gently released from rest. The pendulum starts making oscillations. The time for a number of oscillations (say 20 or 50) is measured with a stop watch and the time period is obtained. The value of g is calculated by equation (12.23). The length of the thread is varied and the experiment is repeated a number of times to minimise the effect of random errors.

Example 12.8

In a laboratory experiment with simple pendulum it was found that it took 36 s *to complete* 20 *oscillations when the effective length was kept at* 80 cm. *Calculate the acceleration due to gravity from these data.*

Solution **:** The time period of a simple pendulum is given by

$$T = 2\pi\sqrt{lg^{-1}}$$

or, $$g = \frac{4\pi^2 l}{T^2}. \quad \text{... (i)}$$

In the experiment described in the question, the time period is

$$T = \frac{36 \text{ s}}{20} = 1{\cdot}8 \text{ s}.$$

Thus, by (i),

$$g = \frac{4\pi^2 \times 0{\cdot}80 \text{ m}}{(1{\cdot}8 \text{ s})^2} = 9{\cdot}75 \text{ m s}^{-2}.$$

12.9 PHYSICAL PENDULUM

Any rigid body suspended from a fixed support constitutes a physical pendulum. A circular ring suspended on a nail in a wall, a heavy metallic rod suspended through a hole in it, etc., are examples of physical pendulum. Figure (12.13) shows a physical pendulum. A rigid body is suspended through a hole at O. When the centre of mass C is vertically below O, the body may remain at rest. We call this position $\theta = 0$. When the body is pulled aside and released, it executes oscillations.

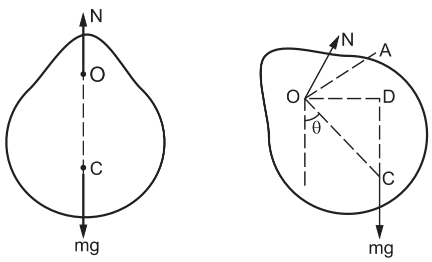

Figure 12.13

The body rotates about a horizontal axis through O and perpendicular to the plane of motion. Let this axis be OA. Suppose the angular displacement of the body is θ at time t. The line OC makes an angle θ with the vertical at this instant.

Forces on the body are (a) the weight mg and (b) the contact force $\mathscr{N}$ by the support at O.

The torque of $\mathscr{N}$ about OA is zero as the force $\mathscr{N}$ acts through the point O. The torque of mg has magnitude

$$|\Gamma| = mg\,(OD)$$
$$= mg\,(OC)\sin\theta = mgl\sin\theta$$

where $l = OC$ is the separation between the point of suspension and the centre of mass. This torque tries to bring the body back towards $\theta = 0$. Thus, we can write

$$\Gamma = -mgl\sin\theta.$$

If the moment of inertia of the body about OA is I, the angular acceleration becomes

$$\alpha = \frac{\Gamma}{I} = -\frac{mgl}{I}\sin\theta. \qquad \dots \text{ (i)}$$

We see that the angular acceleration is not proportional to the angular displacement and the motion is not strictly simple harmonic. However, for small displacements $\sin\theta \approx \theta$ so that equation (i) becomes

$$\alpha = -\omega^2\theta$$

where $\omega^2 = mglI^{-1}$.

Thus, for small oscillations, the motion is nearly simple harmonic. The time period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}}\,. \qquad \dots \text{ (12.24)}$$

Example 12.9

A uniform rod of length 1·00 m *is suspended through an end and is set into oscillation with small amplitude under gravity. Find the time period of oscillation.*

Solution **:** For small amplitude the angular motion is nearly simple harmonic and the time period is given by

$$T = 2\pi\sqrt{\frac{I}{mgl}} = 2\pi\sqrt{\frac{(mL^2/3)}{mgL/2}}$$

$$= 2\pi\sqrt{\frac{2L}{3g}} = 2\pi\sqrt{\frac{2 \times 1{\cdot}00 \text{ m}}{3 \times 9{\cdot}80 \text{ m s}^{-2}}} = 1{\cdot}64 \text{ s.}$$

12.10 TORSIONAL PENDULUM

In torsional pendulum, an extended body is suspended by a light thread or a wire. The body is rotated through an angle about the wire as the axis of rotation (figure 12.14).

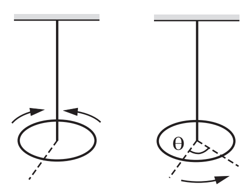

Figure 12.14

The wire remains vertical during this motion but a twist is produced in the wire. The lower end of the wire is rotated through an angle with the body but the upper end remains fixed with the support. Thus, a twist θ is produced. The twisted wire exerts a restoring torque on the body to bring it back to its original position in which the twist θ in the wire is zero. This torque has a magnitude proportional to the angle of twist which is equal to the angle rotated by the body. The proportionality constant is called the *torsional constant* of the wire. Thus, if the torsional constant of the wire is k and the body is rotated through an angle θ, the torque produced is $\Gamma = -k\theta$.

If I be the moment of inertia of the body about the vertical axis, the angular acceleration is

$$\alpha = \frac{\Gamma}{I} = -\frac{k}{I}\theta$$

$$= -\omega^2 \theta$$

where $$\omega = \sqrt{\frac{k}{I}} \cdot$$

Thus, the motion of the body is simple harmonic and the time period is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{k}} \cdot \quad \ldots \text{ (12.25)}$$

Example 12.10

A uniform disc of radius 5·0 cm *and mass* 200 g *is fixed at its centre to a metal wire, the other end of which is fixed with a clamp. The hanging disc is rotated about the wire through an angle and is released. If the disc makes torsional oscillations with time period* 0·20 s, *find the torsional constant of the wire.*

Solution : The situation is shown in figure (12.15). The moment of inertia of the disc about the wire is

$$I = \frac{mr^2}{2} = \frac{(0{\cdot}200 \text{ kg})\,(5{\cdot}0 \times 10^{-2} \text{ m})^2}{2}$$

$$= 2{\cdot}5 \times 10^{-4} \text{ kg-m}^2.$$

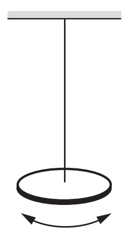

Figure 12.15

The time period is given by

$$T = 2\pi \sqrt{\frac{I}{k}}$$

or, $$k = \frac{4\pi^2 I}{T^2}$$

$$= \frac{4\pi^2 (2{\cdot}5 \times 10^{-4} \text{ kg-m}^2)}{(0{\cdot}20 \text{ s})^2}$$

$$= 0{\cdot}25 \frac{\text{kg-m}^2}{\text{s}^2} \cdot$$

12.11 COMPOSITION OF TWO SIMPLE HARMONIC MOTIONS

A simple harmonic motion is produced when a restoring force proportional to the displacement acts on a particle. If the particle is acted upon by two separate forces each of which can produce a simple harmonic motion, the resultant motion of the particle is a combination of two simple harmonic motions.

Let $\vec{r}_1$ denote the position of the particle at time t if the force $\vec{F}_1$ alone acts on it. Similarly, let $\vec{r}_2$ denote the position at time t if the force $\vec{F}_2$ alone acts on it. Newton's second law gives,

$$m \frac{d^2 \vec{r}_1}{dt^2} = \vec{F}_1$$

and $$m \frac{d^2 \vec{r}_2}{dt^2} = \vec{F}_2.$$

Adding them,

$$m \frac{d^2 \vec{r}_1}{dt^2} + m \frac{d^2 \vec{r}_2}{dt^2} = \vec{F}_1 + \vec{F}_2$$

or, $$m \frac{d^2}{dt^2} (\vec{r}_1 + \vec{r}_2) = \vec{F}_1 + \vec{F}_2. \quad \ldots \text{ (i)}$$

But $\vec{F}_1 + \vec{F}_2$ is the resultant force acting on the particle and so the position $\vec{r}$ of the particle when both the forces act, is given by

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_1 + \vec{F}_2. \quad \ldots \text{ (ii)}$$

Comparing (i) and (ii) we can show that

$$\vec{r} = \vec{r}_1 + \vec{r}_2$$

and $$\vec{u} = \vec{u}_1 + \vec{u}_2$$

if these conditions are met at $t = 0$.

Thus, if two forces $\vec{F}_1$ and $\vec{F}_2$ act together on a particle, its position at any instant can be obtained as follows. Assume that only the force $\vec{F}_1$ acts and find the position $\vec{r}_1$ at that instant. Then assume that only the force $\vec{F}_2$ acts and find the position $\vec{r}_2$ at that same instant. The actual position will be the vector sum of $\vec{r}_1$ and $\vec{r}_2$.

(A) Composition of two Simple Harmonic Motions in Same Direction

Suppose two forces act on a particle, the first alone would produce a simple harmonic motion given by

$$x_1 = A_1 \sin \omega t$$

and the second alone would produce a simple harmonic motion given by

$$x_2 = A_2 \sin(\omega t + \delta).$$

Both the motions are along the x-direction. The amplitudes may be different and their phases differ by δ. Their frequency is assumed to be same. The resultant position of the particle is then given by

$$\begin{aligned} x &= x_1 + x_2 \\ &= A_1 \sin \omega t + A_2 \sin(\omega t + \delta) \\ &= A_1 \sin \omega t + A_2 \sin \omega t \cos \delta + A_2 \cos \omega t \sin \delta \\ &= (A_1 + A_2 \cos \delta) \sin \omega t + (A_2 \sin \delta) \cos \omega t \end{aligned}$$

$$= C\sin\omega t + D\cos\omega t$$

$$= \sqrt{C^2 + D^2}\left[\frac{C}{\sqrt{C^2 + D^2}}\sin\omega t + \frac{D}{\sqrt{C^2 + D^2}}\cos\omega t\right] \quad \dots \text{(i)}$$

where $C = A_1 + A_2\cos\delta$ and $D = A_2\sin\delta$.

Now $\frac{C}{\sqrt{C^2 + D^2}}$ and $\frac{D}{\sqrt{C^2 + D^2}}$ both have magnitudes less than 1 and the sum of their squares is 1. Thus, we can find an angle ε between 0 and 2π such that

$$\sin\varepsilon = \frac{D}{\sqrt{C^2 + D^2}} \text{ and } \cos\varepsilon = \frac{C}{\sqrt{C^2 + D^2}}.$$

Equation (i) then becomes

$$x = \sqrt{C^2 + D^2}\,(\cos\varepsilon\sin\omega t + \sin\varepsilon\cos\omega t)$$

or, $$x = A\sin(\omega t + \varepsilon) \quad \dots (12.26)$$

where

$$A = \sqrt{C^2 + D^2}$$

$$= \sqrt{(A_1 + A_2\cos\delta)^2 + (A_2\sin\delta)^2}$$

$$= \sqrt{A_1^2 + 2A_1A_2\cos\delta + A_2^2\cos^2\delta + A_2^2\sin^2\delta}$$

$$= \sqrt{A_1^2 + 2A_1A_2\cos\delta + A_2^2} \quad \dots (12.27)$$

and $$\tan\varepsilon = \frac{D}{C} = \frac{A_2\sin\delta}{A_1 + A_2\cos\delta}. \quad \dots (12.28)$$

Equation (12.26) shows that the resultant of two simple harmonic motions along the same direction is itself a simple harmonic motion. The amplitude and phase of the resultant simple harmonic motion depend on the amplitudes of the two component simple harmonic motions as well as the phase difference between them.

Amplitude of The Resultant Simple Harmonic Motion

The amplitude of the resultant simple harmonic motion is given by equation (12.27),

$$A = \sqrt{A_1^2 + 2A_1A_2\cos\delta + A_2^2}.$$

If $\delta = 0$, the two simple harmonic motions are in phase

$$A = \sqrt{A_1^2 + 2A_1A_2 + A_2^2} = A_1 + A_2.$$

The amplitude of the resultant motion is equal to the sum of amplitudes of the individual motions. This is the maximum possible amplitude.

If $\delta = \pi$, the two simple harmonic motions are out of phase and

$$A = \sqrt{A_1^2 - 2A_1A_2 + A_2^2} = A_1 - A_2 \text{ or } A_2 - A_1.$$

As the amplitude is always positive we can write $A = |A_1 - A_2|$. If $A_1 = A_2$ the resultant amplitude is zero and the particle does not oscillate at all.

For any value of δ other than 0 and π the resultant amplitude is between $|A_1 - A_2|$ and $A_1 + A_2$.

Example 12.11

Find the amplitude of the simple harmonic motion obtained by combining the motions

$$x_1 = (2{\cdot}0 \text{ cm})\sin\omega t$$

and $$x_2 = (2{\cdot}0 \text{ cm})\sin(\omega t + \pi/3).$$

Solution **:** The two equations given represent simple harmonic motions along *X*-axis with amplitudes A_1 = 2·0 cm and A_2 = 2·0 cm. The phase difference between the two simple harmonic motions is $\pi/3$. The resultant simple harmonic motion will have an amplitude A given by

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta}$$

$$= \sqrt{(2{\cdot}0\text{ cm})^2 + (2{\cdot}0\text{ cm})^2 + 2\,(2{\cdot}0\text{ cm})^2\cos\frac{\pi}{3}}$$

$$= 3{\cdot}5 \text{ cm}.$$

Vector Method of Combining Two Simple Harmonic Motions

There is a very useful method to remember the equations of resultant simple harmonic motion when two simple harmonic motions of same frequency and in same direction combine. Suppose the two individual motions are represented by

$$x_1 = A_1\sin\omega t$$

and $$x_2 = A_2\sin(\omega t + \delta).$$

Let us for a moment represent the first simple harmonic motion by a vector of magnitude A_1 and the second simple harmonic motion by another vector of magnitude A_2. We draw these vectors in figure (12.16). The vector A_2 is drawn at an angle δ with A_1 to represent that the second simple harmonic motion has a phase difference of δ with the first simple harmonic motion.

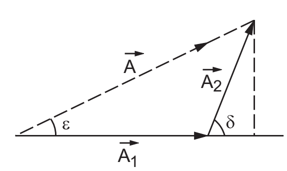

Figure 12.16

The resultant $\vec{A}$ of these two vectors will represent the resultant simple harmonic motion. As we know from vector algebra, the magnitude of the resultant vector is

$$A = \sqrt{A_1^2 + 2A_1A_2\cos\delta + A_2^2}$$

which is same as equation (12.27). The resultant $\vec{A}$ makes an angle ε with $\vec{A}_1$, where

$$\tan\varepsilon = \frac{A_2 \sin\delta}{A_1 + A_2 \cos\delta}$$

which is same as equation (12.28).

This method can easily be extended to more than two vectors. Figure (12.17) shows the construction for adding three simple harmonic motions in the same direction.

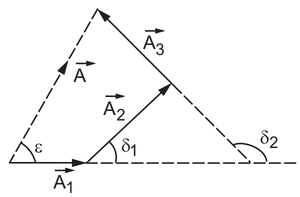

Figure 12.17

$$x_1 = A_1 \sin\omega t$$
$$x_2 = A_2 \sin(\omega t + \delta_1)$$
$$x_3 = A_3 \sin(\omega t + \delta_2).$$

The resultant motion is given by $x = A \sin(\omega t + \varepsilon)$.

(B) Composition of Two Simple Harmonic Motions in Perpendicular Directions

Suppose two forces act on a particle, the first alone would produce a simple harmonic motion in x-direction given by

$$x = A_1 \sin\omega t \qquad \dots \text{(i)}$$

and the second would produce a simple harmonic motion in y-direction given by

$$y = A_2 \sin(\omega t + \delta). \qquad \dots \text{(ii)}$$

The amplitudes A_1 and A_2 may be different and their phases differ by δ. The frequencies of the two simple harmonic motions are assumed to be equal. The resultant motion of the particle is a combination of the two simple harmonic motions. The position of the particle at time t is (x, y) where x is given by equation (i) and y is given by (ii). The motion is thus two-dimensional and the path of the particle is in general an ellipse. The equation of the path may be obtained by eliminating t from (i) and (ii).

By (i),

$$\sin\omega t = \frac{x}{A_1} \cdot$$

Thus, $$\cos\omega t = \sqrt{1 - \frac{x^2}{A_1^2}} \cdot$$

Putting in (ii)

$$y = A_2 [\sin\omega t \cos\delta + \cos\omega t \sin\delta]$$

$$= A_2 \left[\frac{x}{A_1} \cos\delta + \sqrt{1 - \frac{x^2}{A_1^2}} \sin\delta \right]$$

or, $$\left(\frac{y}{A_2} - \frac{x}{A_1} \cos\delta \right)^2 = \left(1 - \frac{x^2}{A_1^2} \right) \sin^2\delta$$

or, $$\frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} \cos\delta + \frac{x^2}{A_1^2} \cos^2\delta$$
$$= \sin^2\delta - \frac{x^2}{A_1^2} \sin^2\delta$$

or, $$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy \cos\delta}{A_1 A_2} = \sin^2\delta. \qquad \dots (12.29)$$

This is an equation of an ellipse and hence the particle moves in ellipse. Equation (i) shows that x remains between $-A_1$ and $+A_1$ and (ii) shows that y remains between A_2 and $-A_2$. Thus, the particle always remains inside the rectangle defined by

$$x = \pm A_1, \; y = \pm A_2 .$$

The ellipse given by (12.29) is traced inside this rectangle and touches it on all the four sides (figure 12.18).

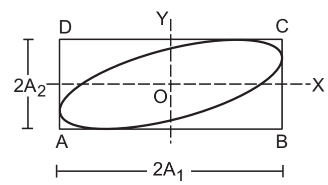

Figure 12.18

Special Cases

(a) $\delta = 0$

The two simple harmonic motions are in phase. When the x-coordinate of the particle crosses the value 0, the y-coordinate also crosses the value 0. When x-coordinate reaches its maximum value A_1, the y-coordinate also reaches its maximum value A_2. Similarly, when x-coordinate reaches its minimum value $-A_1$, the y-coordinate reaches its minimum value $-A_2$.

If we substitute $\delta = 0$ in equation (12.29) we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} = 0$$

or, $$\left(\frac{x}{A_1} - \frac{y}{A_2} \right)^2 = 0$$

or, $$y = \frac{A_2}{A_1} x \qquad \dots \text{(iii)}$$

which is the equation of a straight line passing through the origin and having a slope $\tan^{-1}\frac{A_2}{A_1}\cdot$ Figure (12.19) shows the path. Equation (iii) represents the diagonal AC of the rectangle. The particle moves on this diagonal.

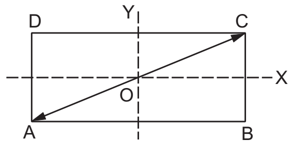

Figure 12.19

Equation (iii) can be directly obtained by dividing (i) by (ii) and putting $\delta = 0$. The displacement of the particle on this straight line at time t is

$$r = \sqrt{x^2 + y^2} = \sqrt{(A_1 \sin\omega t)^2 + (A_2 \sin\omega t)^2}$$

$$= \sqrt{(A_1^2 + A_2^2)}\ \sin\omega t.$$

Thus, the resultant motion is a simple harmonic motion with same frequency and phase as the component motions. The amplitude of the resultant simple harmonic motion is $\sqrt{A_1^2 + A_2^2}$ as is also clear from figure (12.19).

(b) $\delta = \pi$

The two simple harmonic motions are out of phase in this case. When the x-coordinate of the particle reaches its maximum value A_1, the y-coordinate reaches its minimum value $-A_2$. Similarly, when the x-coordinate reaches its minimum value $-A_1$, the y-coordinate takes its maximum value A_2.

Putting $\delta = \pi$ in equation (12.29) we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} + \frac{2xy}{A_1 A_2} = 0$$

or, $$\left(\frac{x}{A_1} + \frac{y}{A_2}\right)^2 = 0$$

or, $$y = -\frac{A_2}{A_1}\cdot x$$

which is the equation of the line BD in figure (12.20).

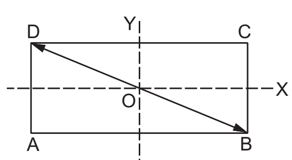

Figure 12.20

Thus the particle oscillates on the diagonal BD of the rectangle as shown in figure (12.20).

The displacement on this line at time t may be obtained from equation (i) and (ii) (with $\delta = \pi$).

$$r = \sqrt{x^2 + y^2} = \sqrt{[A_1 \sin\omega t]^2 + [A_2 \sin(\omega t + \pi)]^2}$$

$$= \sqrt{A_1^2 \sin^2\omega t + A_2^2 \sin^2\omega t} = \sqrt{A_1^2 + A_2^2}\ \sin\omega t.$$

Thus the resultant motion is a simple harmonic motion with amplitude $\sqrt{A_1^2 + A_2^2}$.

(c) $\delta = \pi/2$

The two simple harmonic motions differ in phase by $\pi/2$. Equations (i) and (ii) may be written as

$$x = A_1 \sin\omega t$$
$$y = A_2 \sin(\omega t + \pi/2) = A_2 \cos\omega t.$$

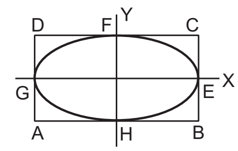

Figure 12.21

The x-coordinate takes its maximum value $x = A_1$ when $\sin\omega t = 1$. Then $\cos\omega t = 0$ and hence, the y-coordinate is zero. The particle is at the point E in figure (12.21). When x-coordinate reduces to 0, $\sin\omega t = 0$, and $\cos\omega t$ becomes 1. Then y-coordinate takes its maximum value A_2 so that the particle reaches the point F. Then x reduces to $-A_1$ and y becomes 0. This corresponds to the point G of figure (12.21). As x increases to 0 again, y takes its minimum value $-A_2$, the particle is at the point H. The motion of the particle is along an ellipse $EFGHE$ inscribed in the rectangle shown. The major and the minor axes of the ellipse are along the X and Y-axes.

Putting $\delta = \pi/2$ in equation (12.29) we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$$

which is the standard equation of an ellipse with its axes along X and Y-axes and with its centre at the origin. The length of the major and minor axes are $2A_1$ and $2A_2$.

If $A_1 = A_2 = A$ together with $\delta = \pi/2$, the rectangle of figure (12.21) becomes a square and the ellipse becomes a circle. Equation (12.29) becomes

$$x^2 + y^2 = A^2$$

which represents a circle.

Thus, the combination of two simple harmonic motions of equal amplitude in perpendicular directions differing in phase by $\pi/2$ is a circular motion.

The circular motion may be clockwise or anticlockwise, depending on which component leads the other.

12.12 DAMPED HARMONIC MOTION

A particle will execute a simple harmonic motion with a constant amplitude if the resultant force on it is proportional to the displacement and is directed opposite to it. Nature provides a large number of situations in which such restoring force acts. The spring-mass system and the simple pendulum are examples. However, in many of the cases some kind of damping force is also present with the restoring force. The damping force may arise due to friction between the moving parts, air resistance or several other causes. The damping force is a function of speed of the moving system and is directed opposite to the velocity. Energy is lost due to the negative work done by the damping force and the system comes to a halt in due course.

The damping force may be a complicated function of speed. In several cases of practical interest the damping force is proportional to the speed. This force may then be written as

$$F = -bv.$$

The equation of motion is

$$m\frac{dv}{dt} = -kx - bv.$$

This equation can be solved using standard methods of calculus. For small damping the solution is of the form

$$x = A_0 e^{-\frac{bt}{2m}} \sin(\omega' t + \delta) \qquad \dots \quad (12.30)$$

where $\omega' = \sqrt{(k/m) - (b/2m)^2} = \sqrt{{\omega_0}^2 - (b/2m)^2}$.

For small b, the angular frequency $\omega' \approx \sqrt{k/m} = \omega_0$. Thus, the system oscillates with almost the natural angular frequency $\sqrt{k/m}$ (with which the system will oscillate if there is no damping) and with amplitude decreasing with time according to the equation

$$A = A_0 e^{-\frac{bt}{2m}}.$$

The amplitude decreases with time and finally becomes zero. Figure (12.22) shows qualitatively the displacement of the particle as a function of time.

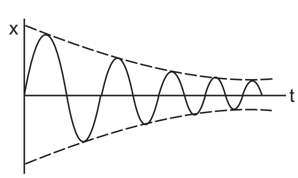

Figure 12.22

If the damping is large the system may not oscillate at all. If displaced, it will go towards the mean position and stay there without overshooting on the other side. The damping for which the oscillation just ceases is called *critical damping*.

12.13 FORCED OSCILLATION AND RESONANCE

In certain situations apart from the restoring force and the damping force, there is yet another force applied on the body which itself changes periodically with time. As a simplest case suppose a force $F = F_0 \sin\omega t$ is applied to a body of mass m on which a restoring force $-kx$ and a damping force $-bv$ is acting. The equation of motion for such a body is

$$m\frac{dv}{dt} = -kx - bv + F_0 \sin\omega t.$$

The motion is somewhat complicated for some time and after this the body oscillates with the frequency ω of the applied periodic force. The displacement is given by

$$x = A\sin(\omega t + \phi).$$

Such an oscillation is called *forced oscillation*. The amplitude of the oscillation is given by

$$A = \frac{F_0/m}{\sqrt{({\omega}^2 - {\omega_0}^2)^2 + (b\omega/m)^2}} \qquad \dots \quad (12.31)$$

where $\omega_0 = \sqrt{k/m}$ is the natural angular frequency.

In forced oscillation the energy lost due to the damping force is compensated by the work done by the applied force. The oscillations with constant amplitude are, therefore, sustained.

If we vary the angular frequency ω of the applied force, this amplitude changes and becomes maximum when $\omega = \omega' = \sqrt{\omega_0^2 - b^2/(2m)^2}$. This condition is called *resonance*. For small damping $\omega' \approx \omega_0$ and the resonance occurs when the applied frequency is (almost) equal to the natural frequency.

Figure (12.23) shows the amplitude as a function of the applied frequency. We see that the amplitude is large if the damping is small. Also the resonance is sharp in this case, that is the amplitude rapidly falls if ω is different from ω_0.

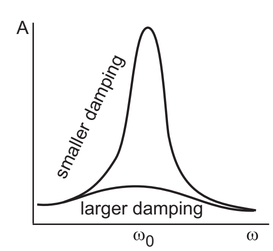

Figure 12.23

If the damping were ideally zero, the amplitude of the forced vibration at resonance would be infinity by equation (12.31). Some damping is always present in mechanical systems and the amplitude remains finite.

However, the amplitude may become very large if the damping is small and the applied frequency is close to the natural frequency. This effect is important in designing bridges and other civil constructions. On July 1, 1940, the newly constructed Tacoma Narrows Bridge (Washington) was opened for traffic. Only four months after this, a mild wind set up the bridge in resonant vibrations. In a few hours the amplitude became so large that the bridge could not stand the stress and a part broke off and went into the water below.

Worked Out Examples

1. *The equation of a particle executing simple harmonic motion is* $x = (5\text{ m})\sin\left[(\pi\text{ s}^{-1})t + \frac{\pi}{3}\right]$. *Write down the amplitude, time period and maximum speed. Also find the velocity at* $t = 1$ s.

Solution : Comparing with equation $x = A\sin(\omega t + \delta)$, we see that

the amplitude = 5 m,

and time period $= \frac{2\pi}{\omega} = \frac{2\pi}{\pi\text{ s}^{-1}} = 2\text{ s}$.

The maximum speed $= A\omega = 5\text{ m} \times \pi\text{ s}^{-1} = 5\pi\text{ m s}^{-1}$.

The velocity at time $t = \frac{dx}{dt} = A\omega\cos(\omega t + \delta)$.

At $t = 1$ s,

$$v = (5\text{ m})(\pi\text{ s}^{-1})\cos\left(\pi + \frac{\pi}{3}\right) = -\frac{5\pi}{2}\text{ m s}^{-1}.$$

2. *A block of mass* 5 kg *executes simple harmonic motion under the restoring force of a spring. The amplitude and the time period of the motion are* 0·1 m *and* 3·14 s *respectively. Find the maximum force exerted by the spring on the block.*

Solution : The maximum force exerted on the block is kA when the block is at the extreme position.

The angular frequency $\omega = \frac{2\pi}{T} = 2\text{ s}^{-1}$.

The spring constant $= k = m\omega^2$

$= (5\text{ kg})(4\text{ s}^{-2}) = 20\text{ N m}^{-1}$.

Maximum force $= kA = (20\text{ N m}^{-1})(0{\cdot}1\text{ m}) = 2\text{ N}$.

3. *A particle executing simple harmonic motion has angular frequency* $6{\cdot}28\text{ s}^{-1}$ *and amplitude* 10 cm. *Find* (*a*) *the time period,* (*b*) *the maximum speed,* (*c*) *the maximum acceleration,* (*d*) *the speed when the displacement is* 6 cm *from the mean position,* (*e*) *the speed at* $t = 1/6$ s *assuming that the motion starts from rest at* $t = 0$.

Solution :

(a) Time period $= \frac{2\pi}{\omega} = \frac{2\pi}{6{\cdot}28}\text{ s} = 1\text{ s}$.

(b) Maximum speed $= A\omega = (0{\cdot}1\text{ m})(6{\cdot}28\text{ s}^{-1})$

$= 0{\cdot}628\text{ m s}^{-1}$.

(c) Maximum acceleration $= A\omega^2$

$= (0{\cdot}1\text{ m})(6{\cdot}28\text{ s}^{-1})^2$

$= 4\text{ m s}^{-2}$.

(d) $v = \omega\sqrt{A^2 - x^2} = (6{\cdot}28\text{ s}^{-1})\sqrt{(10\text{ cm})^2 - (6\text{ cm})^2}$

$= 50{\cdot}2\text{ cm s}^{-1}$.

(e) At $t = 0$, the velocity is zero, i.e., the particle is at an extreme. The equation for displacement may be written as

$$x = A\cos\omega t.$$

The velocity is $v = -A\,\omega\sin\omega t$.

At $t = \frac{1}{6}$ s, $\quad v = -(0{\cdot}1\text{ m})(6{\cdot}28\text{ s}^{-1})\sin\left(\frac{6{\cdot}28}{6}\right)$

$= (-0{\cdot}628\text{ m s}^{-1})\sin\frac{\pi}{3}$

$= -54{\cdot}4\text{ cm s}^{-1}$.

4. *A particle executes a simple harmonic motion of time period T. Find the time taken by the particle to go directly from its mean position to half the amplitude.*

Solution : Let the equation of motion be $x = A\sin\omega t$.

At $t = 0$, $x = 0$ and hence the particle is at its mean position. Its velocity is

$$v = A\,\omega\cos\omega t = A\,\omega$$

which is positive. So it is going towards $x = A/2$.

The particle will be at $x = A/2$, at a time t, where

$$\frac{A}{2} = A\sin\omega t$$

or, $\quad \sin\omega t = 1/2$

or, $\quad \omega t = \pi/6$.

Here minimum positive value of ωt is chosen because we are interested in finding the time taken by the particle to directly go from $x = 0$ to $x = A/2$.

Thus, $t = \frac{\pi}{6\,\omega} = \frac{\pi}{6(2\pi/T)} = \frac{T}{12}$.

5. *A block of mass m hangs from a vertical spring of spring constant k. If it is displaced from its equilibrium position, find the time period of oscillations.*

Solution **:** Suppose the length of the spring is stretched by a length Δl. The tension in the spring is $k\ \Delta l$ and this is the force by the spring on the block. The other force on the block is mg due to gravity. For equilibrium, $mg = k\ \Delta l$ or $\Delta l = mg/k$. Take this position of the block as $x = 0$. If the block is further displaced by x, the resultant force is $k\left(\frac{mg}{k} + x\right) - mg = kx$.

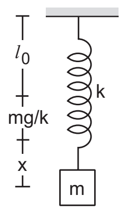

Figure 12-W1

Thus, the resultant force is proportional to the displacement. The motion is simple harmonic with a time period $T = 2\pi\sqrt{\frac{m}{k}}$.

We see that in vertical oscillations, gravity has no effect on time period. The only effect it has is to shift the equilibrium position by a distance mg/k as if the natural length is increased (or decreased if the lower end of the spring is fixed) by mg/k.

6. *A particle suspended from a vertical spring oscillates* 10 *times per second. At the highest point of oscillation the spring becomes unstretched. (a) Find the maximum speed of the block. (b) Find the speed when the spring is stretched by* 0·20 cm. *Take* $g = \pi^2\ \mathrm{m\ s^{-2}}$.

Solution **:**

(a) The mean position of the particle during vertical oscillations is mg/k distance away from its position when the spring is unstretched. At the highest point, i.e., at an extreme position, the spring is unstretched.

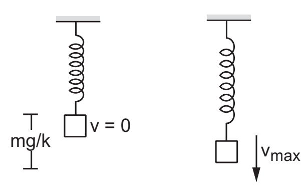

Figure 12-W2

Hence the amplitude is

$$A = \frac{mg}{k}. \qquad \dots \text{(i)}$$

The angular frequency is

$$\omega = \sqrt{\frac{k}{m}} = 2\pi\nu = (20\pi)\ \mathrm{s^{-1}} \qquad \dots \text{(ii)}$$

or, $$\frac{m}{k} = \frac{1}{400\,\pi^2}\ \mathrm{s^2}.$$

Putting in (i), the amplitude is

$$A = \left(\frac{1}{400\,\pi^2}\ \mathrm{s^2}\right)\left(\pi^2\ \frac{\mathrm{m}}{\mathrm{s^2}}\right)$$
$$= \frac{1}{400}\ \mathrm{m} = 0{\cdot}25\ \mathrm{cm}.$$

The maximum speed $= A\,\omega$

$$= (0{\cdot}25\ \mathrm{cm})\,(20\,\pi\ \mathrm{s^{-1}}) = 5\,\pi\ \mathrm{cm\ s^{-1}}.$$

(b) When the spring is stretched by 0·20 cm, the block is 0·25 cm – 0·20 cm = 0·05 cm above the mean position. The speed at this position will be

$$v = \omega\sqrt{A^2 - x^2}$$
$$= (20\,\pi\ \mathrm{s^{-1}})\sqrt{(0{\cdot}25\ \mathrm{cm})^2 - (0{\cdot}05\ \mathrm{cm})^2}$$
$$\cong 15{\cdot}4\ \mathrm{cm\ s^{-1}}.$$

7. *The pulley shown in figure (12-W3) has a moment of inertia I about its axis and mass m. Find the time period of vertical oscillation of its centre of mass. The spring has spring constant k and the string does not slip over the pulley.*

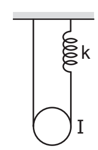

Figure 12-W3

Solution **:** Let us first find the equilibrium position. For rotational equilibrium of the pulley, the tensions in the two strings should be equal. Only then the torque on the pulley will be zero. Let this tension be T. The extension of the spring will be $y = T/k$, as the tension in the spring will be the same as the tension in the string. For translational equilibrium of the pulley,

$$2\,T = mg \ \text{ or, } \ 2\,ky = mg \quad \text{or,} \quad y = \frac{mg}{2\,k}.$$

The spring is extended by a distance $\frac{mg}{2\,k}$ when the pulley is in equilibrium.

Now suppose, the centre of the pulley goes down further by a distance x. The total increase in the length of the string plus the spring is $2x$ (x on the left of the pulley and x on the right). As the string has a constant length, the extension of the spring is $2x$. The energy of the system is

$$U = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 - mgx + \frac{1}{2}k\left(\frac{mg}{2\,k} + 2\,x\right)^2$$

$$= \frac{1}{2}\left(\frac{I}{r^2} + m\right)v^2 + \frac{m^2 g^2}{8k} + 2kx^2.$$

As the system is conservative, $\frac{dU}{dt} = 0$,

giving $\quad 0 = \left(\frac{I}{r^2} + m\right)v\frac{dv}{dt} + 4kxv$

or, $\quad \frac{dv}{dt} = -\frac{4kx}{\left(\frac{I}{r^2} + m\right)}$

or, $\quad a = -\omega^2 x,$ where $\omega^2 = \frac{4k}{\left(\frac{I}{r^2} + m\right)}$.

Thus, the centre of mass of the pulley executes a simple harmonic motion with time period

$$T = 2\pi\sqrt{\left(\frac{I}{r^2} + m\right)/(4k)}\ .$$

8. *The friction coefficient between the two blocks shown in figure (12-W4) is* μ *and the horizontal plane is smooth. (a) If the system is slightly displaced and released, find the time period. (b) Find the magnitude of the frictional force between the blocks when the displacement from the mean position is x. (c) What can be the maximum amplitude if the upper block does not slip relative to the lower block* ?

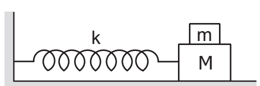

Figure 12-W4

Solution **:**

(a) For small amplitude, the two blocks oscillate together. The angular frequency is

$$\omega = \sqrt{\frac{k}{M+m}}$$

and so the time period $T = 2\pi\sqrt{\frac{M+m}{k}}$.

(b) The acceleration of the blocks at displacement x from the mean position is

$$a = -\omega^2 x = \frac{-kx}{M+m}.$$

The resultant force on the upper block is, therefore,

$$ma = \frac{-mkx}{M+m}.$$

This force is provided by the friction of the lower block. Hence, the magnitude of the frictional force is $\frac{mk\,|x|}{M+m}$.

(c) Maximum force of friction required for simple harmonic motion of the upper block is $\frac{mkA}{M+m}$ at the extreme positions. But the maximum frictional force can only be μ mg. Hence

$$\frac{mkA}{M+m} = \mu mg \text{ or, } A = \frac{\mu(M+m)g}{k}.$$

9. *The left block in figure (12-W5) collides inelastically with the right block and sticks to it. Find the amplitude of the resulting simple harmonic motion.*

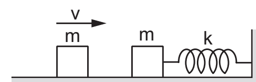

Figure 12-W5

Solution **:** Assuming the collision to last for a small interval only, we can apply the principle of conservation of momentum. The common velocity after the collision is $\frac{v}{2}$. The kinetic energy $= \frac{1}{2}(2m)\left(\frac{v}{2}\right)^2 = \frac{1}{4}mv^2$. This is also the total energy of vibration as the spring is unstretched at this moment. If the amplitude is A, the total energy can also be written as $\frac{1}{2}kA^2$. Thus

$$\frac{1}{2}kA^2 = \frac{1}{4}mv^2, \text{ giving } A = \sqrt{\frac{m}{2k}}\,v.$$

10. *Describe the motion of the mass m shown in figure (*12-W6*). The walls and the block are elastic.*

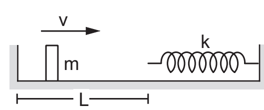

Figure 12-W6

Solution **:** The block reaches the spring with a speed v. It now compresses the spring. The block is decelerated due to the spring force, comes to rest when $\frac{1}{2}mv^2 = \frac{1}{2}kx^2$ and returns back. It is accelerated due to the spring force till the spring acquires its natural length. The contact of the block with the spring is now broken. At this instant it has regained its speed v (towards left) as the spring is unstretched and no potential energy is stored. This process takes half the period of oscillation, i.e., $\pi\sqrt{m/k}$. The block strikes the left wall after a time L/v and as the collision is elastic, it rebounds with the same speed v. After a time L/v, it again reaches the spring and the process is repeated. The block thus undergoes periodic motion with time period $\pi\sqrt{m/k} + \frac{2L}{v}$.

11. *A block of mass m is suspended from the ceiling of a stationary standing elevator through a spring of spring constant k. Suddenly, the cable breaks and the elevator starts falling freely. Show that the block now executes a* *simple harmonic motion of amplitude mg/k in the elevator.*

Solution **:** When the elevator is stationary, the spring is stretched to support the block. If the extension is x, the tension is kx which should balance the weight of the block.

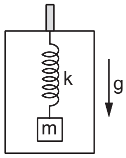

Figure 12-W7

Thus, $x = mg/k$. As the cable breaks, the elevator starts falling with acceleration 'g'. We shall work in the frame of reference of the elevator. Then we have to use a pseudo force mg upward on the block. This force will 'balance' the weight. Thus, the block is subjected to a net force kx by the spring when it is at a distance x from the position of unstretched spring. Hence, its motion in the elevator is simple harmonic with its mean position corresponding to the unstretched spring. Initially, the spring is stretched by $x = mg/k$, where the velocity of the block (with respect to the elevator) is zero. Thus, the amplitude of the resulting simple harmonic motion is mg/k.

12. *The spring shown in figure (12-W8) is kept in a stretched position with extension x_0 when the system is released. Assuming the horizontal surface to be frictionless, find the frequency of oscillation.*

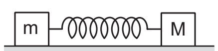

Figure 12-W8

Solution **:** Considering "the two blocks plus the spring" as a system, there is no external resultant force on the system. Hence the centre of mass of the system will remain at rest. The mean positions of the two simple harmonic motions occur when the spring becomes unstretched. If the mass m moves towards right through a distance x and the mass M moves towards left through a distance X before the spring acquires natural length,

$$x + X = x_0. \qquad \dots \text{(i)}$$

x and X will be the amplitudes of the two blocks m and M respectively. As the centre of mass should not change during the motion, we should also have

$$mx = MX. \qquad \dots \text{(ii)}$$

From (i) and (ii), $x = \dfrac{Mx_0}{M+m}$ and $X = \dfrac{mx_0}{M+m}$.

Hence, the left block is $x = \dfrac{Mx_0}{M+m}$ distance away from its mean position in the beginning of the motion. The force by the spring on this block at this instant is equal to the tension of spring, i.e., $T = kx_0$.

Now $x = \dfrac{Mx_0}{M+m}$ or, $x_0 = \dfrac{M+m}{M}x$

Thus, $T = \dfrac{k(M+m)}{M}x$ or, $a = \dfrac{T}{m} = \dfrac{k(M+m)}{Mm}x$.

The angular frequency is, therefore, $\omega = \sqrt{\dfrac{k(M+m)}{Mm}}$

and the frequency is $\nu = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{k(M+m)}{Mm}}$.

13. *Assume that a narrow tunnel is dug between two diametrically opposite points of the earth. Treat the earth as a solid sphere of uniform density. Show that if a particle is released in this tunnel, it will execute a simple harmonic motion. Calculate the time period of this motion.*

Solution **:**

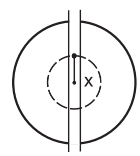

Figure 12-W9

Consider the situation shown in figure (12-W9). Suppose at an instant t the particle in the tunnel is at a distance x from the centre of the earth. Let us draw a sphere of radius x with its centre at the centre of the earth. Only the part of the earth within this sphere will exert a net attraction on the particle. Mass of this part is

$$M' = \frac{\frac{4}{3}\pi x^3}{\frac{4}{3}\pi R^3}M = \frac{x^3}{R^3}M.$$

The force of attraction is, therefore,

$$F = \frac{G(x^3/R^3)\,Mm}{x^2} = \frac{GMm}{R^3}x.$$

This force acts towards the centre of the earth. Thus, the resultant force on the particle is opposite to the displacement from the centre of the earth and is proportional to it. The particle, therefore, executes a simple harmonic motion in the tunnel with the centre of the earth as the mean position.

The force constant is $k = \dfrac{GMm}{R^3}$, so that the time period is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{R^3}{GM}}.$$

14. *A simple pendulum of length* 40 cm *oscillates with an angular amplitude of* 0·04 rad. *Find (a) the time period, (b) the linear amplitude of the bob, (c) the speed of the bob when the string makes* 0·02 rad *with the vertical and (d) the angular acceleration when the bob is in momentary rest. Take* $g = 10 \text{ m s}^{-2}$.

Solution **:**

(a) The angular frequency is

$$\omega = \sqrt{g/l} = \sqrt{\frac{10 \text{ m s}^{-2}}{0{\cdot}4 \text{ m}}} = 5 \text{ s}^{-1}.$$

The time period is

$$\frac{2\pi}{\omega} = \frac{2\pi}{5 \text{ s}^{-1}} = 1{\cdot}26 \text{ s}.$$

(b) Linear amplitude $= 40 \text{ cm} \times 0{\cdot}04 = 1{\cdot}6 \text{ cm}$.

(c) Angular speed at displacement 0·02 rad is

$$\Omega = (5 \text{ s}^{-1}) \sqrt{(0{\cdot}04)^2 - (0{\cdot}02)^2} \text{ rad} = 0{\cdot}17 \text{ rad s}^{-1}.$$

Linear speed of the bob at this instant

$$= (40 \text{ cm}) \times 0{\cdot}17 \text{ s}^{-1} = 6{\cdot}8 \text{ cm s}^{-1}.$$

(d) At momentary rest, the bob is in extreme position. Thus, the angular acceleration

$$\alpha = (0{\cdot}04 \text{ rad})(25 \text{ s}^{-2}) = 1 \text{ rad s}^{-2}.$$

15. *A simple pendulum having a bob of mass m undergoes small oscillations with amplitude* θ_0. *Find the tension in the string as a function of the angle made by the string with the vertical. When is this tension maximum, and when is it minimum?*

Solution : Suppose the speed of the bob at angle θ is v. Using conservation of energy between the extreme position and the position with angle θ,

$$\frac{1}{2} mv^2 = mgl(\cos\theta - \cos\theta_0). \qquad \ldots \text{ (i)}$$

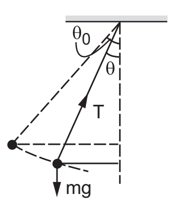

Figure 12-W10

As the bob moves in a circular path, the force towards the centre should be equal to mv^2/l. Thus,

$$T - mg\cos\theta = mv^2/l.$$

Using (i),

$$T - mg\cos\theta = 2\,mg(\cos\theta - \cos\theta_0)$$

or, $$T = 3\,mg\cos\theta - 2\,mg\cos\theta_0.$$

Now $\cos\theta$ is maximum at $\theta = 0$ and decreases as $|\theta|$ increases (for $|\theta| < 90°$).

Thus, the tension is maximum when $\theta = 0$, i.e., at the mean position and is minimum when $\theta = \pm\theta_0$, i.e., at extreme positions.

16. *A simple pendulum is taken at a place where its separation from the earth's surface is equal to the radius of the earth. Calculate the time period of small oscillations if the length of the string is* 1·0 m. *Take* $g = \pi^2 \text{ m s}^{-2}$ *at the surface of the earth.*

Solution **:** At a height R (radius of the earth) the acceleration due to gravity is

$$g' = \frac{GM}{(R+R)^2} = \frac{1}{4}\frac{GM}{R^2} = g/4.$$

The time period of small oscillations of the simple pendulum is

$$T = 2\pi\sqrt{l/g'} = 2\pi\sqrt{\frac{1{\cdot}0 \text{ m}}{\frac{1}{4}\times\pi^2 \text{ m s}^{-2}}} = 2\pi\left(\frac{2}{\pi}\text{ s}\right) = 4 \text{ s}.$$

17. *A simple pendulum is suspended from the ceiling of a car accelerating uniformly on a horizontal road. If the acceleration is* a_0 *and the length of the pendulum is* l, *find the time period of small oscillations about the mean position.*

Solution **:** We shall work in the car frame. As it is accelerated with respect to the road, we shall have to apply a pseudo force ma_0 on the bob of mass m.

For mean position, the acceleration of the bob with respect to the car should be zero. If θ be the angle made by the string with the vertical, the tension, weight and the pseudo force will add to zero in this position.

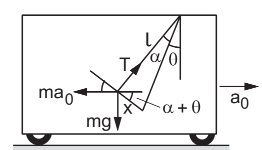

Figure 12-W11

Suppose, at some instant during oscillation, the string is further deflected by an angle α so that the displacement of the bob is x. Taking the components perpendicular to the string,

component of $T = 0$,

component of $mg = mg\sin(\alpha + \theta)$ and

component of $ma_0 = -ma_0\cos(\alpha + \theta)$.

Thus, the resultant component F

$$= m[g\sin(\alpha + \theta) - a_0\cos(\alpha + \theta)].$$

Expanding the sine and cosine and putting $\cos\alpha \approx 1$, $\sin\alpha \approx \alpha = x/l$, we get

$$F = m\left[g\sin\theta - a_0\cos\theta + (g\cos\theta + a_0\sin\theta)\frac{x}{l}\right]. \qquad \ldots \text{ (i)}$$

At $x = 0$, the force F on the bob should be zero, as this is the mean position. Thus by (i),

$$0 = m[g \sin\theta - a_0 \cos\theta] \qquad \dots \text{(ii)}$$

giving $\tan\theta = \frac{a_0}{g}$

Thus, $$\sin\theta = \frac{a_0}{\sqrt{a_0^2 + g^2}} \qquad \dots \text{(iii)}$$

$$\cos\theta = \frac{g}{\sqrt{a_0^2 + g^2}} \cdot \qquad \dots \text{(iv)}$$

Putting (ii), (iii) and (iv) in (i), $F = m\sqrt{g^2 + a_0^2}\,\frac{x}{l}$

or, $F = m\,\omega^2 x$, where $\omega^2 = \frac{\sqrt{g^2 + a_0^2}}{l}$.

This is an equation of simple harmonic motion with time period

$$t = \frac{2\pi}{\omega} = 2\pi \frac{\sqrt{l}}{\left(g^2 + a_0^2\right)^{1/4}} \cdot$$

An easy working rule may be found out as follows. In the mean position, the tension, the weight and the pseudo force balance.

From figure (12-W12), the tension is

$$T = \sqrt{(ma_0)^2 + (mg)^2}$$

or, $$\frac{T}{m} = \sqrt{a_0^2 + g^2}\,.$$

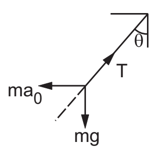

Figure 12-W12

This plays the role of effective 'g'. Thus the time period is

$$t = 2\pi\sqrt{\frac{l}{T/m}} = 2\pi \frac{\sqrt{l}}{[g^2 + a_0^2]^{1/4}} \cdot$$

18. *A uniform meter stick is suspended through a small pin hole at the* 10 cm *mark. Find the time period of small oscillation about the point of suspension.*

Solution **:** Let the mass of the stick be m. The moment of inertia of the stick about the axis of rotation through the point of suspension is

$$I = \frac{ml^2}{12} + md^2,$$

where $l = 1$ m and $d = 40$ cm.

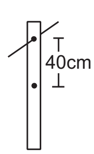

Figure 12-W13

The separation between the centre of mass of the stick and the point of suspension is $d = 40$ cm. The time period of this physical pendulum is

$$T = 2\,\pi\sqrt{\frac{I}{mgd}}$$

$$= 2\pi\sqrt{\left(\frac{ml^2}{12} + md^2\right)/(mgd)}$$

$$= 2\pi\left[\sqrt{\left(\frac{1}{12} + 0{\cdot}16\right)/4}\right] \text{s} = 1{\cdot}55 \text{ s.}$$

19. *The moment of inertia of the disc used in a torsional pendulum about the suspension wire is* 0·2 kg-m^2. *It oscillates with a period of* 2 s. *Another disc is placed over the first one and the time period of the system becomes* 2·5 s. *Find the moment of inertia of the second disc about the wire.*

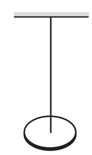
Figure 12-W14

Solution **:**

Let the torsional constant of the wire be k. The moment of inertia of the first disc about the wire is 0·2 kg-m^2. Hence, the time period is

$$2 \text{ s} = 2\pi\sqrt{\frac{I}{K}}$$

$$= 2\pi\sqrt{\frac{0{\cdot}2 \text{ kg-m}^2}{k}} \cdot \qquad \dots \text{(i)}$$

When the second disc having moment of inertia I_1 about the wire is added, the time period is

$$2{\cdot}5 \text{ s} = 2\pi\sqrt{\frac{0{\cdot}2 \text{ kg-m}^2 + I_1}{k}} \qquad \dots \text{(ii)}$$

From (i) and (ii), $\frac{6{\cdot}25}{4} = \frac{0{\cdot}2 \text{ kg-m}^2 + I_1}{0{\cdot}2 \text{ kg-m}^2}$.

This gives $I_1 \approx 0{\cdot}11$ kg-m^2.

20. *A uniform rod of mass m and length l is suspended through a light wire of length l and torsional constant k as shown in figure (12-W15). Find the time period if the system makes (a) small oscillations in the vertical plane about the suspension point and (b) angular oscillations in the horizontal plane about the centre of the rod.*

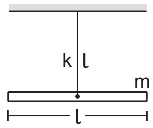

Figure 12-W15

Solution :

(a) The oscillations take place about the horizontal line through the point of suspension and perpendicular to the plane of the figure. The moment of inertia of the rod about this line is

$$\frac{ml^2}{12} + ml^2 = \frac{13}{12} ml^2.$$

$$\text{The time period} = 2\pi \sqrt{\frac{I}{mgl}} = 2\pi \sqrt{\frac{13\, ml^2}{12\, mgl}}$$

$$= 2\pi \sqrt{\frac{13\, l}{12\, g}}\,.$$

(b) The angular oscillations take place about the suspension wire. The moment of inertia about this line is $ml^2/12$. The time period is

$$2\pi \sqrt{\frac{I}{k}} = 2\pi \sqrt{\frac{ml^2}{12\, k}}\,.$$

21. *A particle is subjected to two simple harmonic motions*

$$x_1 = A_1 \sin\omega t$$

and $\quad x_2 = A_2 \sin(\omega t + \pi/3)$.

Find (a) the displacement at $t = 0$, *(b) the maximum speed of the particle and (c) the maximum acceleration of the particle.*

Solution :

(a) At $t = 0$, $\quad x_1 = A_1 \sin\omega t = 0$

and $x_2 = A_2 \sin(\omega t + \pi/3)$

$$= A_2 \sin(\pi/3) = \frac{A_2\sqrt{3}}{2}\,.$$

Thus, the resultant displacement at $t = 0$ is

$$x = x_1 + x_2 = \frac{A_2\sqrt{3}}{2}\,.$$

(b) The resultant of the two motions is a simple harmonic motion of the same angular frequency ω. The amplitude of the resultant motion is

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos(\pi/3)}$$

$$= \sqrt{A_1^2 + A_2^2 + A_1A_2}\,.$$

The maximum speed is

$$v_{max} = A\,\omega = \omega\sqrt{A_1^2 + A_2^2 + A_1A_2}\,.$$

(c) The maximum acceleration is

$$a_{max} = A\,\omega^2 = \omega^2\sqrt{A_1^2 + A_2^2 + A_1A_2}\,.$$

22. *A particle is subjected to two simple harmonic motions in the same direction having equal amplitudes and equal frequency. If the resultant amplitude is equal to the amplitude of the individual motions, find the phase difference between the individual motions.*

Solution : Let the amplitudes of the individual motions be A each. The resultant amplitude is also A. If the phase difference between the two motions is δ,

$$A = \sqrt{A^2 + A^2 + 2A\,.\,A\,.\cos\delta}$$

or, $$= A\sqrt{2(1+\cos\delta)} = 2A\cos\frac{\delta}{2}$$

or, $$\cos\frac{\delta}{2} = \frac{1}{2}$$

or, $$\delta = 2\pi/3.$$

□

QUESTIONS FOR SHORT ANSWER

1. A person goes to bed at sharp 10·00 pm every day. Is it an example of periodic motion ? If yes, what is the time period ? If no, why ?

2. A particle executing simple harmonic motion comes to rest at the extreme positions. Is the resultant force on the particle zero at these positions according to Newton's first law ?

3. Can simple harmonic motion take place in a noninertial frame? If yes, should the ratio of the force applied with the displacement be constant ?

4. A particle executes simple harmonic motion. If you are told that its velocity at this instant is zero, can you say what is its displacement ? If you are told that its velocity at this instant is maximum, can you say what is its displacement ?

5. A small creature moves with constant speed in a vertical circle on a bright day. Does its shadow formed by the sun on a horizontal plane move in a simple harmonic motion ?

6. A particle executes simple harmonic motion. Let P be a point near the mean position and Q be a point near an extreme. The speed of the particle at P is larger than the speed at Q. Still the particle crosses P and Q equal number of times in a given time interval. Does it make you unhappy ?

7. In measuring time period of a pendulum, it is advised to measure the time between consecutive passage through the mean position in the same direction. This is said to result in better accuracy than measuring time between consecutive passage through an extreme position. Explain.

8. It is proposed to move a particle in simple harmonic motion on a rough horizontal surface by applying an external force along the line of motion. Sketch the graph of the applied force against the position of the particle. Note that the applied force has two values for a given position depending on whether the particle is moving in positive or negative direction.

9. Can the potential energy in a simple harmonic motion be negative ? Will it be so if we choose zero potential energy at some point other than the mean position ?

10. The energy of a system in simple harmonic motion is given by $E = \frac{1}{2} m \omega^2 A^2$. Which of the following two statements is more appropriate ?
(A) The energy is increased because the amplitude is increased.
(B) The amplitude is increased because the energy is increased.

11. A pendulum clock gives correct time at the equator. Will it gain time or loose time as it is taken to the poles ?

12. Can a pendulum clock be used in an earth-satellite ?

13. A hollow sphere filled with water is used as the bob of a pendulum. Assume that the equation for simple pendulum is valid with the distance between the point of suspension and centre of mass of the bob acting as the effective length of the pendulum. If water slowly leaks out of the bob, how will the time period vary ?

14. A block of known mass is suspended from a fixed support through a light spring. Can you find the time period of vertical oscillation only by measuring the extension of the spring when the block is in equilibrium ?

15. A platoon of soldiers marches on a road in steps according to the sound of a marching band. The band is stopped and the soldiers are ordered to break the steps while crossing a bridge. Why ?

16. The force acting on a particle moving along X-axis is $F = -k(x - v_0 t)$ where k is a positive constant. An observer moving at a constant velocity v_0 along the X-axis looks at the particle. What kind of motion does he find for the particle ?

OBJECTIVE I

1. A student says that he had applied a force $F = -k\sqrt{x}$ on a particle and the particle moved in simple harmonic motion. He refuses to tell whether k is a constant or not. Assume that he has worked only with positive x and no other force acted on the particle.
(a) As x increases k increases.
(b) As x increases k decreases.
(c) As x increases k remains constant.
(d) The motion cannot be simple harmonic.

2. The time period of a particle in simple harmonic motion is equal to the time between consecutive appearances of the particle at a particular point in its motion. This point is
(a) the mean position (b) an extreme position
(c) between the mean position and the positive extreme
(d) between the mean position and the negative extreme.

3. The time period of a particle in simple harmonic motion is equal to the smallest time between the particle acquiring a particular velocity $\vec{v}$. The value of v is
(a) v_{max} (b) 0
(c) between 0 and v_{max} (d) between 0 and $-v_{max}$.

4. The displacement of a particle in simple harmonic motion in one time period is
(a) A (b) $2A$ (c) $4A$ (d) zero.

5. The distance moved by a particle in simple harmonic motion in one time period is
(a) A (b) $2A$ (c) $4A$ (d) zero.

6. The average acceleration in one time period in a simple harmonic motion is
(a) $A\omega^2$ (b) $A\omega^2/2$ (c) $A\omega^2/\sqrt{2}$ (d) zero.

7. The motion of a particle is given by $x = A \sin\omega t + B \cos\omega t$. The motion of the particle is
(a) not simple harmonic
(b) simple harmonic with amplitude $A + B$
(c) simple harmonic with amplitude $(A + B) / 2$
(d) simple harmonic with amplitude $\sqrt{A^2 + B^2}$.

8. The displacement of a particle is given by $\vec{r} = A(\vec{i} \cos\omega t + \vec{j} \sin\omega t)$. The motion of the particle is
(a) simple harmonic (b) on a straight line
(c) on a circle (d) with constant acceleration.

9. A particle moves on the X-axis according to the equation $x = A + B \sin\omega t$. The motion is simple harmonic with amplitude
(a) A (b) B (c) $A + B$ (d) $\sqrt{A^2 + B^2}$.

10. Figure (12-Q1) represents two simple harmonic motions.

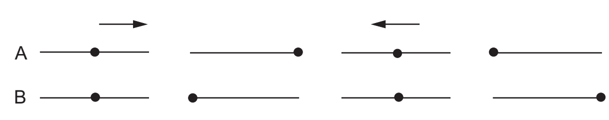

Figure 12-Q1

The parameter which has different values in the two motions is
(a) amplitude (b) frequency
(c) phase (d) maximum velocity.

11. The total mechanical energy of a spring-mass system in simple harmonic motion is $E = \frac{1}{2} m \omega^2 A^2$. Suppose the oscillating particle is replaced by another particle of double the mass while the amplitude A remains the same. The new mechanical energy will
(a) become $2E$ (b) become $E/2$
(c) become $\sqrt{2}E$ (d) remain E.

12. The average energy in one time period in simple harmonic motion is
(a) $\frac{1}{2} m \omega^2 A^2$ (b) $\frac{1}{4} m \omega^2 A^2$
(c) $m \omega^2 A^2$ (d) zero.

13. A particle executes simple harmonic motion with a frequency ν. The frequency with which the kinetic energy oscillates is
(a) $\nu/2$ (b) ν (c) 2ν (d) zero.

14. A particle executes simple harmonic motion under the restoring force provided by a spring. The time period is T. If the spring is divided in two equal parts and one part is used to continue the simple harmonic motion, the time period will
(a) remain T (b) become $2T$
(c) become $T/2$ (d) become $T/\sqrt{2}$.

15. Two bodies A and B of equal mass are suspended from two separate massless springs of spring constant k_1 and k_2 respectively. If the bodies oscillate vertically such that their maximum velocities are equal, the ratio of the amplitude of A to that of B is
(a) k_1/k_2 (b) $\sqrt{k_1/k_2}$
(c) k_2/k_1 (d) $\sqrt{k_2/k_1}$.

16. A spring-mass system oscillates with a frequency ν. If it is taken in an elevator slowly accelerating upward, the frequency will
(a) increase (b) decrease
(c) remain same (d) become zero.

17. A spring-mass system oscillates in a car. If the car accelerates on a horizontal road, the frequency of oscillation will
(a) increase (b) decrease
(c) remain same (d) become zero.

18. A pendulum clock that keeps correct time on the earth is taken to the moon. It will run
(a) at correct rate (b) 6 times faster
(c) $\sqrt{6}$ times faster (d) $\sqrt{6}$ times slower.

19. A wall clock uses a vertical spring-mass system to measure the time. Each time the mass reaches an extreme position, the clock advances by a second. The clock gives correct time at the equator. If the clock is taken to the poles it will
(a) run slow (b) run fast
(c) stop working (d) give correct time.

20. A pendulum clock keeping correct time is taken to high altitudes,
(a) it will keep correct time
(b) its length should be increased to keep correct time
(c) its length should be decreased to keep correct time
(d) it cannot keep correct time even if the length is changed.

21. The free end of a simple pendulum is attached to the ceiling of a box. The box is taken to a height and the pendulum is oscillated. When the bob is at its lowest point, the box is released to fall freely. As seen from the box during this period, the bob will
(a) continue its oscillation as before
(b) stop
(c) will go in a circular path
(d) move on a straight line.

OBJECTIVE II

1 Select the correct statements.
(a) A simple harmonic motion is necessarily periodic.
(b) A simple harmonic motion is necessarily oscillatory.
(c) An oscillatory motion is necessarily periodic.
(d) A periodic motion is necessarily oscillatory.

2. A particle moves in a circular path with a uniform speed. Its motion is
(a) periodic (b) oscillatory
(c) simple harmonic (d) angular simple harmonic.

3. A particle is fastened at the end of a string and is whirled in a vertical circle with the other end of the string being fixed. The motion of the particle is
(a) periodic (b) oscillatory
(c) simple harmonic (d) angular simple harmonic.

4. A particle moves in a circular path with a continuously increasing speed. Its motion is
(a) periodic (b) oscillatory
(c) simple harmonic (d) none of them.

5. The motion of a torsional pendulum is
(a) periodic (b) oscillatory
(c) simple harmonic (d) angular simple harmonic.

6. Which of the following quantities are always negative in a simple harmonic motion ?
(a) $\vec{F} \cdot \vec{a}$. (b) $\vec{v} \cdot \vec{r}$. (c) $\vec{a} \cdot \vec{r}$. (d) $\vec{F} \cdot \vec{r}$.

7. Which of the following quantities are always positive in a simple harmonic motion ?
(a) $\vec{F} \cdot \vec{a}$. (b) $\vec{v} \cdot \vec{r}$. (c) $\vec{a} \cdot \vec{r}$. (d) $\vec{F} \cdot \vec{r}$.

8. Which of the following quantities are always zero in a simple harmonic motion ?
(a) $\vec{F} \times \vec{a}$. (b) $\vec{v} \times \vec{r}$. (c) $\vec{a} \times \vec{r}$. (d) $\vec{F} \times \vec{r}$.

9. Suppose a tunnel is dug along a diameter of the earth. A particle is dropped from a point, a distance h directly

above the tunnel. The motion of the particle as seen from the earth is
(a) simple harmonic (b) parabolic
(c) on a straight line (d) periodic.

10. For a particle executing simple harmonic motion, the acceleration is proportional to
(a) displacement from the mean position
(b) distance from the mean position
(c) distance travelled since $t = 0$
(d) speed.

11. A particle moves in the *X-Y* plane according to the equation
$$\vec{r} = (\vec{i} + 2\vec{j})\, A \cos\omega t.$$
The motion of the particle is
(a) on a straight line (b) on an ellipse
(c) periodic (d) simple harmonic.

12. A particle moves on the *X*-axis according to the equation $x = x_0 \sin^2 \omega t$. The motion is simple harmonic
(a) with amplitude x_0 (b) with amplitude $2x_0$
(c) with time period $\frac{2\pi}{\omega}$ (d) with time period $\frac{\pi}{\omega}$.

13. In a simple harmonic motion
(a) the potential energy is always equal to the kinetic energy
(b) the potential energy is never equal to the kinetic energy
(c) the average potential energy in any time interval is equal to the average kinetic energy in that time interval
(d) the average potential energy in one time period is equal to the average kinetic energy in this period.

14. In a simple harmonic motion
(a) the maximum potential energy equals the maximum kinetic energy
(b) the minimum potential energy equals the minimum kinetic energy
(c) the minimum potential energy equals the maximum kinetic energy
(d) the maximum potential energy equals the minimum kinetic energy.

15. An object is released from rest. The time it takes to fall through a distance h and the speed of the object as it falls through this distance are measured with a pendulum clock. The entire apparatus is taken on the moon and the experiment is repeated
(a) the measured times are same
(b) the measured speeds are same
(c) the actual times in the fall are equal
(d) the actual speeds are equal.

16. Which of the following will change the time period as they are taken to moon ?
(a) A simple pendulum (b) A physical pendulum
(c) A torsional pendulum (d) A spring-mass system

EXERCISES

1. A particle executes simple harmonic motion with an amplitude of 10 cm and time period 6 s. At $t = 0$ it is at position $x = 5$ cm going towards positive x-direction. Write the equation for the displacement x at time t. Find the magnitude of the acceleration of the particle at $t = 4$ s.

2. The position, velocity and acceleration of a particle executing simple harmonic motion are found to have magnitudes 2 cm, 1 m s^{-1} and 10 m s^{-2} at a certain instant. Find the amplitude and the time period of the motion.

3. A particle executes simple harmonic motion with an amplitude of 10 cm. At what distance from the mean position are the kinetic and potential energies equal ?

4. The maximum speed and acceleration of a particle executing simple harmonic motion are 10 cm s^{-1} and 50 cm s^{-2}. Find the position(s) of the particle when the speed is 8 cm s^{-1}.

5. A particle having mass 10 g oscillates according to the equation $x = (2{\cdot}0 \text{ cm}) \sin[(100 \text{ s}^{-1})t + \pi/6]$. Find (a) the amplitude, the time period and the spring constant (b) the position, the velocity and the acceleration at $t = 0$.

6. The equation of motion of a particle started at $t = 0$ is given by $x = 5 \sin(20t + \pi/3)$, where x is in centimetre and t in second. When does the particle
(a) first come to rest
(b) first have zero acceleration
(c) first have maximum speed ?

7. Consider a particle moving in simple harmonic motion according to the equation
$$x = 2{\cdot}0 \cos(50\pi t + \tan^{-1} 0{\cdot}75)$$
where x is in centimetre and t in second. The motion is started at $t = 0$. (a) When does the particle come to rest for the first time ? (b) When does the acceleration have its maximum magnitude for the first time ? (c) When does the particle come to rest for the second time ?

8. Consider a simple harmonic motion of time period T. Calculate the time taken for the displacement to change value from half the amplitude to the amplitude.

9. The pendulum of a clock is replaced by a spring-mass system with the spring having spring constant $0{\cdot}1 \text{ N m}^{-1}$. What mass should be attached to the spring ?

10. A block suspended from a vertical spring is in equilibrium. Show that the extension of the spring equals the length of an equivalent simple pendulum, i.e., a pendulum having frequency same as that of the block.

11. A block of mass 0·5 kg hanging from a vertical spring executes simple harmonic motion of amplitude 0·1 m and time period 0·314 s. Find the maximum force exerted by the spring on the block.

12. A body of mass 2 kg suspended through a vertical spring executes simple harmonic motion of period 4 s. If the oscillations are stopped and the body hangs in equilibrium, find the potential energy stored in the spring.

13. A spring stores 5 J of energy when stretched by 25 cm. It is kept vertical with the lower end fixed. A block fastened to its other end is made to undergo small oscillations. If the block makes 5 oscillations each second, what is the mass of the block ?

14. A small block of mass m is kept on a bigger block of mass M which is attached to a vertical spring of spring constant k as shown in the figure. The system oscillates vertically. (a) Find the resultant force on the smaller block when it is displaced through a distance x above its equilibrium position. (b) Find the normal force on the smaller block at this position. When is this force smallest in magnitude ? (c) What can be the maximum amplitude with which the two blocks may oscillate together ?

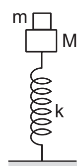

Figure 12-E1

15. The block of mass m_1 shown in figure (12-E2) is fastened to the spring and the block of mass m_2 is placed against it. (a) Find the compression of the spring in the equilibrium position. (b) The blocks are pushed a further distance $(2/k)\ (m_1 + m_2)g \sin\theta$ against the spring and released. Find the position where the two blocks separate. (c) What is the common speed of blocks at the time of separation ?

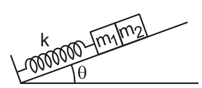

Figure 12-E2

16. In figure (12-E3) $k = 100\ \text{N m}^{-1}$, $M = 1$ kg and $F = 10$ N. (a) Find the compression of the spring in the equilibrium position. (b) A sharp blow by some external agent imparts a speed of 2 m s^{-1} to the block towards left. Find the sum of the potential energy of the spring and the kinetic energy of the block at this instant. (c) Find the time period of the resulting simple harmonic motion. (d) Find the amplitude. (e) Write the potential energy of the spring when the block is at the left extreme. (f) Write the potential energy of the spring when the block is at the right extreme.
The answers of (b), (e) and (f) are different. Explain why this does not violate the principle of conservation of energy.

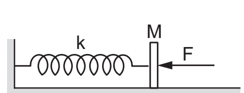

Figure 12-E3

17. Find the time period of the oscillation of mass m in figures 12-E4 a, b, c. What is the equivalent spring constant of the pair of springs in each case ?

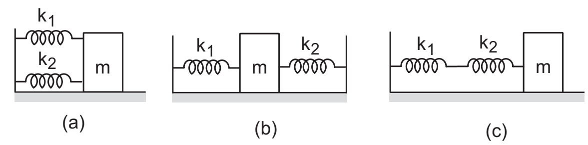

Figure 12-E4

18. The spring shown in figure (12-E5) is unstretched when a man starts pulling on the cord. The mass of the block is M. If the man exerts a constant force F, find (a) the amplitude and the time period of the motion of the block, (b) the energy stored in the spring when the block passes through the equilibrium position and (c) the kinetic energy of the block at this position.

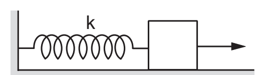

Figure 12-E5

19. A particle of mass m is attatched to three springs A, B and C of equal force constants k as shown in figure (12-E6). If the particle is pushed slightly against the spring C and released, find the time period of oscillation.

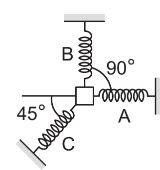

Figure 12-E6

20. Repeat the previous exercise if the angle between each pair of springs is 120° initially.

21. The springs shown in the figure (12-E7) are all unstretched in the beginning when a man starts pulling the block. The man exerts a constant force F on the block. Find the amplitude and the frequency of the motion of the block.

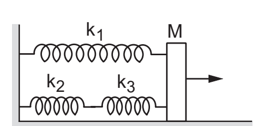

Figure 12-E7

22. Find the elastic potential energy stored in each spring shown in figure (12-E8), when the block is in equilibrium. Also find the time period of vertical oscillation of the block.

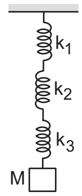

Figure 12-E8

23. The string, the spring and the pulley shown in figure (12-E9) are light. Find the time period of the mass m.

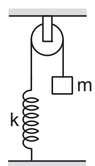

Figure 12-E9

24. Solve the previous problem if the pulley has a moment of inertia I about its axis and the string does not slip over it.

25. Consider the situation shown in figure (12-E10). Show that if the blocks are displaced slightly in opposite directions and released, they will execute simple harmonic motion. Calculate the time period.

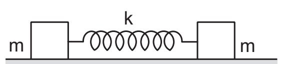

Figure 12-E10

26. A rectangular plate of sides a and b is suspended from a ceiling by two parallel strings of length L each (figure 12-E11). The separation between the strings is d. The plate is displaced slightly in its plane keeping the strings tight. Show that it will execute simple harmonic motion. Find the time period.

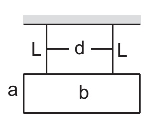

Figure 12-E11

27. A 1 kg block is executing simple harmonic motion of amplitude 0·1 m on a smooth horizontal surface under the restoring force of a spring of spring constant 100 N m^{-1}. A block of mass 3 kg is gently placed on it at the instant it passes through the mean position. Assuming that the two blocks move together, find the frequency and the amplitude of the motion.

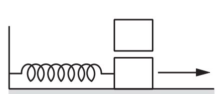

Figure 12-E12

28. The left block in figure (12-E13) moves at a speed v towards the right block placed in equilibrium. All collisions to take place are elastic and the surfaces are frictionless. Show that the motions of the two blocks are periodic. Find the time period of these periodic motions. Neglect the widths of the blocks.

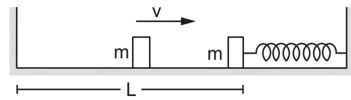

Figure 12-E13

29. Find the time period of the motion of the particle shown in figure (12-E14). Neglect the small effect of the bend near the bottom.

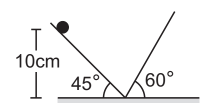

Figure 12-E14

30. All the surfaces shown in figure (12-E15) are frictionless. The mass of the car is M, that of the block is m and the spring has spring constant k. Initially, the car and the block are at rest and the spring is stretched through a length x_0 when the system is released. (a) Find the amplitudes of the simple harmonic motion of the block and of the car as seen from the road. (b) Find the time period(s) of the two simple harmonic motions.

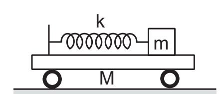

Figure 12-E15

31. A uniform plate of mass M stays horizontally and symmetrically on two wheels rotating in opposite directions (figure 12-E16). The separation between the wheels is L. The friction coefficient between each wheel and the plate is μ. Find the time period of oscillation of the plate if it is slightly displaced along its length and released.

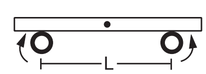

Figure 12-E16

32. A pendulum having time period equal to two seconds is called a seconds pendulum. Those used in pendulum clocks are of this type. Find the length of a seconds pendulum at a place where $g = \pi^2$ m s^{-2}.

33. The angle made by the string of a simple pendulum with the vertical depends on time as $\theta = \frac{\pi}{90} \sin[(\pi \text{ s}^{-1})\, t]$. Find the length of the pendulum if $g = \pi^2$ m s^{-2}.

34. The pendulum of a certain clock has time period 2·04 s. How fast or slow does the clock run during 24 hours ?

35. A pendulum clock giving correct time at a place where $g = 9{\cdot}800 \text{ m s}^{-2}$ is taken to another place where it loses 24 seconds during 24 hours. Find the value of g at this new place.

36. A simple pendulum is constructed by hanging a heavy ball by a 5·0 m long string. It undergoes small oscillations. (a) How many oscillations does it make per second ? (b) What will be the frequency if the system is taken on the moon where acceleration due to gravitation of the moon is $1{\cdot}67 \text{ m s}^{-2}$?

37. The maximum tension in the string of an oscillating pendulum is double of the minimum tension. Find the angular amplitude.

38. A small block oscillates back and forth on a smooth concave surface of radius R (figure 12-E17). Find the time period of small oscillation.

Figure 12-E17

39. A spherical ball of mass m and radius r rolls without slipping on a rough concave surface of large radius R. It makes small oscillations about the lowest point. Find the time period.

40. A simple pendulum of length 40 cm is taken inside a deep mine. Assume for the time being that the mine is 1600 km deep. Calculate the time period of the pendulum there. Radius of the earth = 6400 km.

41. Assume that a tunnel is dug across the earth (radius = R) passing through its centre. Find the time a particle takes to cover the length of the tunnel if (a) it is projected into the tunnel with a speed of $\sqrt{gR}$ (b) it is released from a height R above the tunnel (c) it is thrown vertically upward along the length of tunnel with a speed of $\sqrt{gR}$.

42. Assume that a tunnel is dug along a chord of the earth, at a perpendicular distance $R/2$ from the earth's centre where R is the radius of the earth. The wall of the tunnel is frictionless. (a) Find the gravitational force exerted by the earth on a particle of mass m placed in the tunnel at a distance x from the centre of the tunnel. (b) Find the component of this force along the tunnel and perpendicular to the tunnel. (c) Find the normal force exerted by the wall on the particle. (d) Find the resultant force on the particle. (e) Show that the motion of the particle in the tunnel is simple harmonic and find the time period.

43. A simple pendulum of length l is suspended through the ceiling of an elevator. Find the time period of small oscillations if the elevator (a) is going up with an acceleration a_0 (b) is going down with an acceleration a_0 and (c) is moving with a uniform velocity.

44. A simple pendulum of length 1 feet suspended from the ceiling of an elevator takes $\pi/3$ seconds to complete one oscillation. Find the acceleration of the elevator.

45. A simple pendulum fixed in a car has a time period of 4 seconds when the car is moving uniformly on a horizontal road. When the accelerator is pressed, the time period changes to 3·99 seconds. Making an approximate analysis, find the acceleration of the car.

46. A simple pendulum of length l is suspended from the ceiling of a car moving with a speed v on a circular horizontal road of radius r. (a) Find the tension in the string when it is at rest with respect to the car. (b) Find the time period of small oscillation.

47. The ear-ring of a lady shown in figure (12-E18) has a 3 cm long light suspension wire. (a) Find the time period of small oscillations if the lady is standing on the ground. (b) The lady now sits in a merry-go-round moving at 4 m s^{-1} in a circle of radius 2 m. Find the time period of small oscillations of the ear-ring.

Figure 12-E18

48. Find the time period of small oscillations of the following systems. (a) A metre stick suspended through the 20 cm mark. (b) A ring of mass m and radius r suspended through a point on its periphery. (c) A uniform square plate of edge a suspended through a corner. (d) A uniform disc of mass m and radius r suspended through a point $r/2$ away from the centre.

49. A uniform rod of length l is suspended by an end and is made to undergo small oscillations. Find the length of the simple pendulum having the time period equal to that of the rod.

50. A uniform disc of radius r is to be suspended through a small hole made in the disc. Find the minimum possible time period of the disc for small oscillations. What should be the distance of the hole from the centre for it to have minimum time period ?

51. A hollow sphere of radius 2 cm is attached to an 18 cm long thread to make a pendulum. Find the time period of oscillation of this pendulum. How does it differ from the time period calculated using the formula for a simple pendulum ?

52. A closed circular wire hung on a nail in a wall undergoes small oscillations of amplitude 2^0 and time period 2 s. Find (a) the radius of the circular wire, (b) the speed of the particle farthest away from the point of suspension as it goes through its mean position, (c) the acceleration of this particle as it goes through its mean position and (d) the acceleration of this particle when it is at an extreme position. Take $g = \pi^2 \text{ m s}^{-2}$.

53. A uniform disc of mass m and radius r is suspended through a wire attached to its centre. If the time period of the torsional oscillations be T, what is the torsional constant of the wire?

54. Two small balls, each of mass m are connected by a light rigid rod of length L. The system is suspended from its centre by a thin wire of torsional constant k. The rod is rotated about the wire through an angle θ_0 and released. Find the force exerted by the rod on one of the balls as the system passes through the mean position.

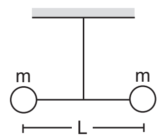

Figure 12-E19

55. A particle is subjected to two simple harmonic motions of same time period in the same direction. The amplitude of the first motion is 3·0 cm and that of the second is 4·0 cm. Find the resultant amplitude if the phase difference between the motions is (a) 0°, (b) 60°, (c) 90°.

56. Three simple harmonic motions of equal amplitudes A and equal time periods in the same direction combine. The phase of the second motion is 60° ahead of the first and the phase of the third motion is 60° ahead of the second. Find the amplitude of the resultant motion.

57. A particle is subjected to two simple harmonic motions given by

$$x_1 = 2{\cdot}0 \sin(100\,\pi\, t) \text{ and } x_2 = 2{\cdot}0 \sin(120\,\pi\, t + \pi/3),$$

where x is in centimeter and t in second. Find the displacement of the particle at (a) $t = 0{\cdot}0125$, (b) $t = 0{\cdot}025$.

58. A particle is subjected to two simple harmonic motions, one along the X-axis and the other on a line making an angle of 45° with the X-axis. The two motions are given by

$$x = x_0 \sin\omega t \text{ and } s = s_0 \sin\omega t$$

Find the amplitude of the resultant motion.

□

ANSWERS

OBJECTIVE I

1. (a)	2. (b)	3. (a)	4. (d)	5. (c)	6. (d)
7. (d)	8. (c)	9. (b)	10. (c)	11. (d)	12. (a)
13. (c)	14. (d)	15. (d)	16. (c)	17. (c)	18. (d)
19. (d)	20. (c)	21. (c).			

OBJECTIVE II

1. (a), (b)	2. (a)	3. (a)
4. (d)	5. (a), (b), (d)	6. (c), (d)
7. (a)	8. all	9. (c), (d)
10. (a)	11. (a), (c), (d)	12. (d)
13. (d)	14. (a), (b)	15. (a), (b)
16. (a), (b).		

EXERCISES

1. $x = (10\text{ cm}) \sin\left(\frac{2\,\pi}{6\text{ s}} t + \frac{\pi}{6}\right)$, $\approx 11\text{ cm s}^{-2}$
2. 4·9 cm, 0·28 s
3. $5\sqrt{2}$ cm
4. ± 1·2 cm from the mean position
5. (a) 2·0 cm, 0·063 s, 100 N m^{-1}

   (b) 1·0 cm, 1·73 m s^{-1}, 100 m s^{-2}
6. (a) $\frac{\pi}{120}$ s (b) $\frac{\pi}{30}$ s (c) $\frac{\pi}{30}$ s
7. (a) $1{\cdot}6 \times 10^{-2}$ s (b) $1{\cdot}6 \times 10^{-2}$ s (c) $3{\cdot}6 \times 10^{-2}$ s
8. $T/6$
9. ≈ 10 g
11. 25 N
12. 40 J
13. 0·16 kg
14. (a) $\frac{mkx}{M+m}$ (b) $mg - \frac{mkx}{M+m}$, at the highest point

    (c) $g\,\frac{(M+m)}{k}$
15. (a) $\frac{(m_1+m_2)g \sin\theta}{k}$

    (b) When the spring acquires its natural length

    (c) $\sqrt{\frac{3}{k}(m_1+m_2)}\; g \sin\theta$
16. (a) 10 cm (b) 2·5 J (c) $\pi/5$ s

    (d) 20 cm (e) 4·5 J (f) 0·5 J
17. (a) $2\pi\sqrt{\frac{m}{k_1+k_2}}$ (b) $2\pi\sqrt{\frac{m}{k_1+k_2}}$ (c) $2\pi\sqrt{\frac{m(k_1+k_2)}{k_1k_2}}$
18. (a) $\frac{F}{k}$, $2\,\pi\sqrt{\frac{M}{k}}$, (b) $\frac{F^2}{2\,k}$ (c) $\frac{F^2}{2\,k}$
19. $2\,\pi\sqrt{\frac{m}{2\,k}}$
20. $2\,\pi\sqrt{\frac{2\,m}{3\,k}}$

21. $\dfrac{F(k_2+k_3)}{k_1k_2+k_2k_3+k_3k_1}$, $\dfrac{1}{2\pi}\sqrt{\dfrac{k_1k_2+k_2k_3+k_3k_1}{M(k_2+k_3)}}$

22. $\dfrac{M^2g^2}{2k_1}$, $\dfrac{M^2g^2}{2k_2}$ and $\dfrac{M^2g^2}{2k_3}$ from above, time period

$$= 2\pi\sqrt{M\left(\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}\right)}$$

23. $2\pi\sqrt{\dfrac{m}{k}}$

24. $2\pi\sqrt{\dfrac{(m+I/r^2)}{k}}$

25. $2\pi\sqrt{\dfrac{m}{2k}}$

26. $2\pi\sqrt{\dfrac{L}{g}}$

27. $\dfrac{5}{2\pi}$ Hz, 5 cm

28. $\left(\pi\sqrt{\dfrac{m}{k}}+\dfrac{2L}{v}\right)$

29. $\approx 0{\cdot}73$ s

30. (a) $\dfrac{Mx_0}{M+m}$, $\dfrac{mx_0}{M+m}$ (b) $2\pi\sqrt{\dfrac{mM}{k(M+m)}}$

31. $2\pi\sqrt{\dfrac{L}{2\mu g}}$

32. 1 m

33. 1 m

34. 28·3 minutes slow

35. 9·795 m s^{-2}

36. (a) $0{\cdot}70/\pi$ (b) $1/(2\pi\sqrt{3})$ Hz

37. $\cos^{-1}(3/4)$

38. $2\pi\sqrt{R/g}$

39. $2\pi\sqrt{\dfrac{7(R-r)}{5g}}$

40. 1·47 s

41. $\dfrac{\pi}{2}\sqrt{\dfrac{R}{g}}$ in each case

42. (a) $\dfrac{GMm}{R^3}\sqrt{x^2+R^2/4}$ (b) $\dfrac{GMm}{R^3}x$, $\dfrac{GMm}{2R^2}$

(c) $\dfrac{GMm}{2R^2}$, (d) $\dfrac{GMm}{R^3}x$ (e) $2\pi\sqrt{R^3/(GM)}$

43. (a) $2\pi\sqrt{\dfrac{l}{g+a_0}}$ (b) $2\pi\sqrt{\dfrac{l}{g-a_0}}$ (c) $2\pi\sqrt{\dfrac{l}{g}}$

44. 4 f s^{-2} upwards

45. $g/10$

46. (a) ma (b) $2\pi\sqrt{l/a}$, where $a=\left[g^2+\dfrac{v^4}{r^2}\right]^{1/2}$

47. (a) 0·34 s (b) 0·30 s

48. (a) 1·51 s (b) $2\pi\sqrt{\dfrac{2r}{g}}$ (c) $2\pi\sqrt{\dfrac{\sqrt{8}\,a}{3g}}$ (d) $2\pi\sqrt{\dfrac{3r}{2g}}$

49. $2l/3$

50. $2\pi\sqrt{\dfrac{r\sqrt{2}}{g}}$, $r/\sqrt{2}$

51. 0·89 s, it is about 0·3% larger than the calculated value

52. (a) 50 cm (b) 11 cm s^{-1}

(c) 1·2 cm s^{-2} towards the point of suspension

(d) 34 cm s^{-2} towards the mean position

53. $\dfrac{2\pi^2mr^2}{T^2}$

54. $\left[\dfrac{k^2\theta_0^4}{L^2}+m^2g^2\right]^{1/2}$

55. (a) 7·0 cm (b) 6·1 cm (c) 5·0 cm

56. $2A$

57. (a) – 2·41 cm (b) 0·27 cm

58. $\left[x_0^2+s_0^2+\sqrt{2}\,x_0s_0\right]^{1/2}$

□