Matrices & Determinants

$$A = \begin{pmatrix} \end{pmatrix}_{m \times n}$$

$m \rightarrow$ rows

$n \rightarrow$ columns

$a_{ij} \rightarrow i^{th}$ row & j^{th} column

$A = (a_{ij})_{m \times n}$ $1 \leq i \leq m$

$1 \leq j \leq n$

sometimes $a_{ij} = f(ij)$ may be given

Ex. $A = (a_{ij})_{3 \times 2}$ $a_{ij} = i + j$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}_{3 \times 2}$$

Minor Matrix & Co-factors for a square Matrix.

$$A = (a_{ij})_{m \times n}$$

$a_{ij} \rightarrow$ Minor matrix

$$\begin{pmatrix} \end{pmatrix}_{(n-1) \times (n-1)}$$

Remaining elements excluding i^{th} row & j^{th} column.

Cofactor-

$a_{ij} \rightarrow C_{ij} = (-1)^{i+j} |M_{ij}|$

Ex.
$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & -1 & 0 \\ 0 & 2 & 5 \end{bmatrix}$$

$$M_{12} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$$

$$C_{12} = (-1)^3 (20-0)$$

$$= -20$$

Determinants

Determinants of a matrix is sum of product of elemenst of any row/.Column with its own cofactors

✗→ ~~Product of~~ Sum of product of elements of any {row/column with the cofactor of any other row/column is zero.

$$A = (a_{ij})_{n \times n}$$
$$|A| = \Sigma(a_{ij} C_{ij})$$

Properties of Determinants

1)
$$Dr = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ g & e & f \end{vmatrix}$$

$$\sum_{r=1}^{n} Dr = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ g & e & f \end{vmatrix}$$

→ It is only applicable when other two rows having **all** constant elements.

(2) → If we interchange **consecutive** rows / column in a determinat, determinant is multiplied by -1. If total no. of changes are m then determinant is multiplied by $(-1)^m$

$$\begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix} \longrightarrow \begin{vmatrix} R_3 \\ R_2 \\ R_1 \end{vmatrix}$$

This change is not directly done.

Process-

$$\begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix} ① \begin{vmatrix} R_2 \\ R_1 \\ R_3 \end{vmatrix} \xrightarrow{②} \begin{vmatrix} R_2 \\ R_3 \\ R_1 \end{vmatrix} ③ \begin{vmatrix} R_3 \\ R_2 \\ R_1 \end{vmatrix}$$

Total change → 3

multiply by $(-1)^3 = -1$.

(3) $D = \begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ d & e & f \end{vmatrix}$

then $\dfrac{d}{dx} D = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ a & b & c \\ d & e & f \end{vmatrix}$

$\int D\, dx = \begin{vmatrix} \int f(x)\, dx & \int g(x)\, dx & \int h(x)\, dx \\ a & b & c \\ d & e & f \end{vmatrix}$

—

Only if other two rows have all constant elements.

If $D = \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix}$

$\dfrac{d}{dx} D = \begin{vmatrix} f_1' & g_1' & h_1' \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2' & g_2' & h_2' \\ f_3 & g_3 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3' & g_3' & h_3' \end{vmatrix}$

$\int D\, dx$ in this case will be calculated by simplifying determinants first. 2nd method is not applicable for integration.

$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - (a^3 + b^3 + c^3)$

$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a) = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

\# $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^7 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

\# $\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$ $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

\# $\begin{vmatrix} 1 & a & a^4 \\ 1 & b & b^4 \\ 1 & c & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$

\# $\begin{vmatrix} 1 & b+c & bc \\ 1 & c+a & ac \\ 1 & a+b & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

\# $|AB| = 0$ It means at least one of the
$|A|$ or $|B|$ is equal to zero.
$|A| = 0$ or $|B| = 0$

\# <u>Multiplication of determinant</u>

$\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 \end{vmatrix}$

$= \begin{vmatrix} 1 \cdot a_1^2 - 2b_1 a_1 + 1 \cdot b_1^2 & 1 \cdot a_1^2 - 2b_1 a_1 + 1 \cdot b_2^2 & \cdot - \\ 1 \cdot a_2^2 - 2b_1 a_2 + 1 \cdot b_1^2 & 1 \cdot a_2^2 - 2b_2 a_2 + 1 \cdot b_2^2 & - \\ 1 \cdot a_3^2 - 2b_1 a_3 + 1 \cdot b_1^2 & 1 \cdot a_3^2 - 2b_2 a_3 + 1 \cdot b_2^2 & - \end{vmatrix}$

$= \begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -2b_1 & b_1^2 \\ 1 & -2b_2 & b_2^2 \\ 1 & -2b_3 & b_3^2 \end{vmatrix}$

Types of Matrix -

$$A = (a_{ij})_{m \times n}$$

→ Square Matrix -

$$m = n$$

$$A = (a_{ij})_{n \times n}$$

→ $m \neq n$

vertical matrix → $m > n$

$$\begin{bmatrix} \\ \\ \end{bmatrix}$$

→ horizontal matrix → $m < n$ $\begin{bmatrix} \end{bmatrix}$

→ Row matrix $m = 1$ $\begin{bmatrix} \end{bmatrix}$

Column matrix $n = 1$ $\begin{bmatrix} \\ \end{bmatrix}$

Diagonal matrix - $a_{ij} = \begin{cases} 0 & i \neq j \\ \in R & i = j \end{cases}$

Scalar matrix - $a_{ij} = \begin{cases} 0 & i \neq j \\ k & i = j \end{cases}$

Identity matrix - $a_{ij} = \begin{cases} 0 & i \neq j \\ 1 & j = j \end{cases} = I_n$

Null / zero Matrix - $a_{ij} = \begin{cases} 0 & i \neq j \\ 0 & i = j \end{cases} = O_n$

Upper triangular matrix - $a_{ij} = \begin{cases} 0 & i > j \\ \in R & i \leq j \end{cases}$ $\begin{bmatrix} \bullet & \cdot \\ 0 & \bullet \\ 0 & 0 \end{bmatrix}$

lower triangular matrix - $a_{ij} = \begin{cases} 0 & i < j \\ \in R & i \geq j \end{cases}$

Equal matrix -

$$A = (a_{ij})_{m \times n}, \quad B = (b_{ij})_{p \times q}$$

$\Rightarrow$ order same; i.e. $m=p$ & $n=q$

$\Rightarrow$ $a_{ij} = b_{ij}$

       then $A = B$

$\longrightarrow$ Two matrix of same order then. $A \pm B$ is valid.

                                        $\in a_{ij} \pm b_{ij}$

$\longrightarrow$ $KA = (Ka_{ij})_{m \times n}$

$\longrightarrow$ Any matrix can be written as sum of upper & lower triangular matrix

$\longrightarrow$ $|A| = 0$ Singular matrix

       $|A| \neq 0$ Non-singular matrix

$\longrightarrow$ $|AB| = 0$

       then at least one of the matrix is singular

\# Multiplication of Matrix-

         $A = (a_{ij})_{m \times n}$

         $B = (b_{ij})_{p \times q}$

         AB is defined   when $n = p$ (order $\rightarrow m \times q$)

         BA is defined   when $q = p$ m   (order $\rightarrow p \times n$)

         $C = AB = (c_{ij})_{m \times q}$ when $n = p$

                $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ $\begin{cases} 1 \leq i \leq m \\ 1 \leq j \leq q \end{cases}$

\# $A + B = B + A$

\# $A(BC) = (AB)c$

\# $AB \neq BA$

\# $A(B \pm C) = AB \pm AC$

\# $AI = IA = A$

If $AB = BA$

$$\cancel{\#\#\#}\ (\lambda A + \mu B)^n = {}^nC_0 (\lambda A)^n + {}^nC_1 (\lambda A)^{n-1}(\mu B) + \cdots \cdots {}^nC_n (\mu B)^n$$

$(I + A)^n = {}^nC_0 I + {}^nC_1 A + {}^nC_2 A^2 + \cdots \cdots {}^nC_n A^n$

$A^n = \underbrace{A \cdot A \cdot A \cdots A}_{n\ times}$ $(n \in N)$

Transpose Matrix -

$$A = (a_{ij})_{m \times n}$$
$$A^T = (a_{ji})_{n \times m}$$

$A, B \rightarrow$ two square matrix, then-

$\rightarrow$ $(A^T)^T = A$

$\rightarrow$ $(A \pm B)^T = A^T \pm B^T$

$\rightarrow$ $(AB)^T = B^T A^T$

$\rightarrow$ $(A^T)^n = (A^n)^T$ $\{A^T\ or\ A' \rightarrow$ trampose of $A\}$

$\rightarrow$ $(A^{-1})^n = (A^n)^{-1}$

$\rightarrow$ $\det(A) = \det(A^T)$

$\rightarrow$ $(\lambda A)^T = \lambda A^T$, $\lambda \in R$

If A is Diagonal matrix,

$$A^T = A$$

Determinant of A -

A is square matrix

$(A^T) = |A|$

$|\lambda A| = \lambda^n |A|$ $n \rightarrow order$

$|AB| = |A||B|$

$|A^n| = |A|^n$

$|A^{-1}| = \dfrac{1}{|A|}$

$|I| = 1$

\# If $f(x) = x^2 - 2x + 3$

& A is any square matrix.
then $f(A) = A^2 - 2A + \underline{3I}$

\#   If $A^T = A \longrightarrow$ Symmetric Matrix $\Rightarrow a_{ij} = a_{ji}$

$A^T = -A \longrightarrow$ Skew Symmetric Matrix

All Diagonal elements of a skew symmetric matrix are zero.

\# Determinant of odd order skew symmetric Matrix is zero.

✷ Any matrix (square matrix) A can be written as the sum of symmetric & skew symmetric matrix.

$$A = \underbrace{\frac{A + A'}{2}}_{Symmetric} + \underbrace{\frac{A - A'}{2}}_{Skew\ Symmetric}$$

\# A square matrix is said to be orthogonal when

→ $A \cdot A' = A' \cdot A = I$

→ $|A| = \pm 1$

→ $A^{-1} = A'$

\# A square matrix is said to be idempotent matrix when $A^2 = A$

then,   → $|A| = 0, 1$

   → $A^n = A$

→ $(I+A)^n = I + \left({}^nC_1 + {}^nC_2 + \_ {}^nC_n \right) A$

$$= I + (2^n - 1) A$$

Involutary Matrix - A ma' square matrix i's said to be
   involutary matrix when $A^2 = I$, then
                      → $|A| = \pm 1$
                      → $A^{-1} = A$
                      → $A^n = \begin{cases} A & n \in odd \\ I & n \in even \end{cases}$

If $A^k = A$ then $k-1$ is period of matrix

Trace of a Matrix -
   Sum of diagonal elements.

$$Tr(A \pm B) = Tr(A) \pm Tr(B)$$
$$Tr(A^T) = Tr(A)$$
$$Tr(\lambda A) = \lambda Tr(A)$$

Adjoint of a Matrix

$$adj(A) = (C)^T$$
$$C = (C_{ij})_{m \times n}$$

$$C_{ij} = (-1)^{i+j} |M_{ij}|$$

Ex. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Nilpotent Matrix -
   $A^m = 0$
   $A^{m-1} \neq 0$
   then A → nilpotent matrix of index m.

If $A = dig(a_1, a_2, a_3 \cdots \cdots a_n)_{n \times n}$

then $adj(A) = dig(a_2 a_3 \cdots a_n, a_1 a_3 \cdots a_n, \cdots \cdots, a_1 a_2 \cdots a_{n-1})_{n \times n}$

Ex:- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow adj(A) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\longrightarrow \quad adj(A^T) = (adj(A))^T$

$\longrightarrow \quad adj(\lambda A) = \lambda^{n-1} adj(A) \quad \lambda \in R$

$\longrightarrow \quad adj(AB) = adj(B) \cdot adj(A)$

$\longrightarrow \quad adj(A^n) = (adj(A))^n \qquad (n \in N)$

$A = dig(a_1, a_2, a_3 \cdots a_n)_{n \times n}$
$B = dig(b_1, b_2, b_3 \cdots b_n)_{n \times n}$

then $\quad AB = dig(a_1 b_1, a_2 b_2, a_3 b_3, \cdots \cdots, a_n b_n)_{n \times n}$

$A \cdot adj(A) = adj(A) \cdot A = |A| I$

Inverse of a Matrix -

For existences of A^{-1}, A is non singular.

$\longrightarrow \quad |A| \neq 0$

$\longrightarrow$ If $AB = BA = I$

then $A^{-1} = B$

or $\quad B^{-1} = A$

$A \cdot adj(A) = adj(A) \cdot A = |A| I.$

So $A \cdot \left\{ \dfrac{adj(A)}{|A|} \right\} = \left\{ \dfrac{adj(A)}{|A|} \right\} \cdot A = I$

$A^{-1} = \dfrac{1}{|A|} \cdot adj(A)$

$(adj(A))^{-1} = \dfrac{A}{|A|}$

$A \cdot adj(A) = |A| I$

$|A| |adj(A)| = |A|^n |I|$

$|adj(A)| = |A|^{n-1}$

$\left|\text{adj}\left(\text{adj}(A)\right)\right| = |A|^{(n-1)^2}$ $n \rightarrow$ order of matrix

$\left|\text{adj}\left(\text{adj}\left(\text{adj} \cdots \underset{k\ times}{\text{adj}(A)}\right)\right)\right| = |A|^{(n-1)^k}$

$\text{adj}\left(\text{adj}(A)\right) = |A|^{n-2} \cdot A$ where $A = (a_{ij})_{n \times n}$

     $\text{adj}\left(\text{adj}(A)\right) = \left|\text{adj}(A)\right| \cdot \left(\text{adj}(A)\right)^{-1}$

           $= |A|^{n-1} \cdot \dfrac{A}{|A|}$

           $= |A|^{n-2} \cdot A$

$|A^{-1}| = \dfrac{1}{|A|}$ $A \cdot A^{-1} = I$

                            $|A| \, |A^{-1}| = 1$

                            $|A^{-1}| = \dfrac{1}{|A|}$

$(\lambda A)^{-1} = \dfrac{1}{\lambda} A^{-1}$

$(A^T)^{-1} = (A^{-1})^T$

   $(AB)^{-1} = B^{-1} A^{-1}$

   $(A^p)^{-1} = (A^{-1})^p$ $p \in N$

If $A = \text{dig.}(a_1, a_2, \cdots a_n)_{n \times n}$

      then $A^{-1} = \text{dig}\left(\dfrac{1}{a_1}, \dfrac{1}{a_2}, \cdots \dfrac{1}{a_n}\right)_{n \times n}$

      Ex. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

           $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

⭐   Every matrix satisfy its characteristic eqⁿ.

$A = (a_{ij})_{n \times n}$

then characteristic eq^n $f(\lambda) = \det (A - \lambda I) = 0$

$\downarrow A$

$f(A) = 0$

Ex. $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ if $A^{-1} = aI + bA$

$a + b = ?$

$(A - \lambda I) = \begin{bmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{bmatrix}$

$= 8 - 6\lambda + \lambda^2 - 3$

$f(\lambda) = \lambda^2 - 6\lambda + 5 = 0$

$f(A) = A^2 - 6A + 5I = 0$

$A^2 - 6A = -5I$

$\dfrac{6A - A^2}{5} = I$

$A \left(\dfrac{6I - A}{5} \right) = I$

$\boxed{A^{-1} = \dfrac{6I - A}{5}}$

so $a = \dfrac{6}{5}$, $b = \dfrac{-1}{5}$

Theory of Eq^n.

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Cramer's Rule -

$$x = \frac{\Delta x}{\Delta} \quad, \quad y = \frac{\Delta y}{\Delta} \quad, \quad z = \frac{\Delta z}{\Delta}$$

(i) if $\Delta \neq 0$ unique soln

(ii) if $\Delta = 0$, $\Delta x = \Delta y = \Delta z = 0$ infinitely many soln

(iii) if $\Delta = 0$, ~~———~~ $\left(\begin{array}{c} \text{at least one} \\ \text{of } \Delta x, \Delta y, \Delta z \neq 0 \end{array}\right)$ no soln

Method-2

$AX = B$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\Rightarrow$ unique soln $|A| \neq 0$

$$X = A^{-1} B$$

infinitely many soln — $|A| = 0$ & $adj(A) \cdot B = 0$

No soln $\rightarrow$ $|A| = 0$, $adj(A) \cdot B \neq 0$

Homogeneous eqⁿ -

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$
$$a_3 x + b_3 y + c_3 z = 0$$

Unique soln   infinitely many soln
$(0,0,0)$
$\Delta \neq 0$ $\Delta = 0$
$|A| \neq 0$ $|A| = 0$

trivial soln   Non-trivial soln

$\rightarrow$ In case of ∞ soln, for finding out value of x, y, z -
     let $x = d$ then put it in two eqⁿ & then
     find out y, z in terms of d.

$\rightarrow$ unique or infinite soln $\longrightarrow$ Consistent
                No soln $\longrightarrow$ ~~Not~~ Non Consistent.