Chapter Two

ELECTROSTATIC POTENTIAL AND CAPACITANCE

2.1 Introduction

In Chapters 6 and 8 (Class XI), the notion of potential energy was introduced. When an external force does work in taking a body from a point to another against a force like spring force or gravitational force, that work gets stored as potential energy of the body. When the external force is removed, the body moves, gaining kinetic energy and losing an equal amount of potential energy. The sum of kinetic and potential energies is thus conserved. Forces of this kind are called conservative forces. Spring force and gravitational force are examples of conservative forces.

Coulomb force between two (stationary) charges is also a conservative force. This is not surprising, since both have inverse-square dependence on distance and differ mainly in the proportionality constants – the masses in the gravitational law are replaced by charges in Coulomb's law. Thus, like the potential energy of a mass in a gravitational field, we can define electrostatic potential energy of a charge in an electrostatic field.

Consider an electrostatic field **E** due to some charge configuration. First, for simplicity, consider the field **E** due to a charge Q placed at the origin. Now, imagine that we bring a test charge q from a point R to a point P against the repulsive force on it due to the charge Q. With reference

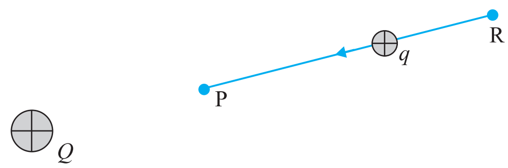

FIGURE 2.1 A test charge q (> 0) is moved from the point R to the point P against the repulsive force on it by the charge Q (> 0) placed at the origin.

to Fig. 2.1, this will happen if Q and q are both positive or both negative. For definiteness, let us take Q, $q > 0$.

Two remarks may be made here. First, we assume that the test charge q is so small that it does not disturb the original configuration, namely the charge Q at the origin (or else, we keep Q fixed at the origin by some unspecified force). Second, in bringing the charge q from R to P, we apply an external force $\mathbf{F}_{ext}$ just enough to counter the repulsive electric force $\mathbf{F}_E$ (i.e, $\mathbf{F}_{ext} = -\mathbf{F}_E$). This means there is no net force on or acceleration of the charge q when it is brought from R to P, i.e., it is brought with infinitesimally slow constant speed. In this situation, work done by the external force is the negative of the work done by the electric force, and gets fully stored in the form of potential energy of the charge q. If the external force is removed on reaching P, the electric force will take the charge away from Q – the stored energy (potential energy) at P is used to provide kinetic energy to the charge q in such a way that the sum of the kinetic and potential energies is conserved.

Thus, work done by external forces in moving a charge q from R to P is

$$W_{RP} = \int_R^P \mathbf{F}_{ext} \cdot d\mathbf{r}$$

$$= -\int_R^P \mathbf{F}_E \cdot d\mathbf{r} \qquad (2.1)$$

This work done is against electrostatic repulsive force and gets stored as potential energy.

At every point in electric field, a particle with charge q possesses a certain electrostatic potential energy, this work done increases its potential energy by an amount equal to potential energy difference between points R and P.

Thus, potential energy difference

$$\Delta U = U_P - U_R = W_{RP} \qquad (2.2)$$

(Note here that this displacement is in an opposite sense to the electric force and hence work done by electric field is negative, i.e., $-W_{RP}$.)

Therefore, we can define electric potential energy difference between two points as the work required to be done by an external force in moving (without accelerating) charge q from one point to another for electric field of any arbitrary charge configuration.

Two important comments may be made at this stage:

(i) The right side of Eq. (2.2) depends only on the initial and final positions of the charge. It means that the work done by an electrostatic field in moving a charge from one point to another depends only on the initial and the final points and is independent of the path taken to go from one point to the other. This is the fundamental characteristic of a conservative force. The concept of the potential energy would not be meaningful if the work depended on the path. The path-independence of work done by an electrostatic field can be proved using the Coulomb's law. We omit this proof here.

(ii) Equation (2.2) defines *potential energy difference* in terms of the physically meaningful quantity *work*. Clearly, potential energy so defined is undetermined to within an additive constant.What this means is that the actual value of potential energy is not physically significant; it is only the difference of potential energy that is significant. We can always add an arbitrary constant α to potential energy at every point, since this will not change the potential energy difference:

$$(U_P + \alpha) - (U_R + \alpha) = U_P - U_R$$

Put it differently, there is a freedom in choosing the point where potential energy is zero. A convenient choice is to have electrostatic potential energy zero at infinity. With this choice, if we take the point R at infinity, we get from Eq. (2.2)

$$W_{\infty P} = U_P - U_\infty = U_P \qquad (2.3)$$

Since the point P is arbitrary, Eq. (2.3) provides us with a definition of potential energy of a charge q at any point. *Potential energy of charge q at a point* (in the presence of field due to any charge configuration) *is the work done by the external force* (equal and opposite to the electric force) *in bringing the charge q from infinity to that point.*

Count Alessandro Volta (1745 – 1827) Italian physicist, professor at Pavia. Volta established that the *animal electricity* observed by Luigi Galvani, 1737–1798, in experiments with frog muscle tissue placed in contact with dissimilar metals, was not due to any exceptional property of animal tissues but was also generated whenever any wet body was sandwiched between dissimilar metals. This led him to develop the first *voltaic pile*, or battery, consisting of a large stack of moist disks of cardboard (electrolyte) sandwiched between disks of metal (electrodes).

COUNT ALESSANDRO VOLTA (1745 – 1827)

2.2 Electrostatic Potential

Consider any general static charge configuration. We define potential energy of a test charge q in terms of the work done on the charge q. This work is obviously proportional to q, since the force at any point is $q\mathbf{E}$, where $\mathbf{E}$ is the electric field at that point due to the given charge configuration. It is, therefore, convenient to divide the work by the amount of charge q, so that the resulting quantity is independent of q. In other words, work done per unit test charge is characteristic of the electric field associated with the charge configuration. This leads to the idea of electrostatic potential V due to a given charge configuration. From Eq. (2.1), we get:

Work done by external force in bringing a unit positive charge from point R to P

$$= V_P - V_R \left(= \frac{U_P - U_R}{q} \right) \qquad (2.4)$$

where V_P and V_R are the electrostatic potentials at P and R, respectively. Note, as before, that it is not the actual value of potential but the potential difference that is physically significant. If, as before, we choose the potential to be zero at infinity, Eq. (2.4) implies:

Work done by an external force in bringing a unit positive charge from infinity to a point = electrostatic potential (V) at that point.

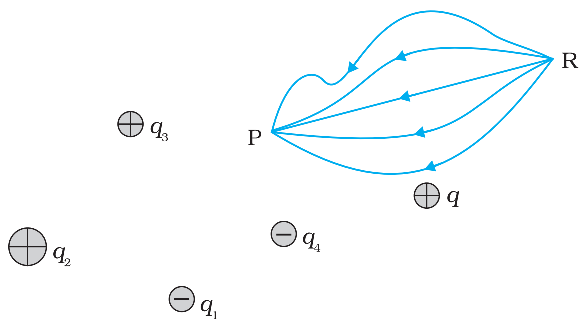

FIGURE 2.2 Work done on a test charge q by the electrostatic field due to any given charge configuration is independent of the path, and depends only on its initial and final positions.

In other words, the electrostatic potential (V) at any point in a region with electrostatic field is the work done in bringing a unit positive charge (without acceleration) from infinity to that point.

The qualifying remarks made earlier regarding potential energy also apply to the definition of potential. To obtain the work done per unit test charge, we should take an infinitesimal test charge δq, obtain the work done δW in bringing it from infinity to the point and determine the ratio $\delta W/\delta q$. Also, the external force at every point of the path is to be equal and opposite to the electrostatic force on the test charge at that point.

2.3 Potential due to a Point Charge

Consider a point charge Q at the origin (Fig. 2.3). For definiteness, take Q to be positive. We wish to determine the potential at any point P with position vector **r** from the origin. For that we must calculate the work done in bringing a unit positive test charge from infinity to the point P. For $Q > 0$, the work done against the repulsive force on the test charge is positive. Since work done is independent of the path, we choose a convenient path – along the radial direction from infinity to the point P.

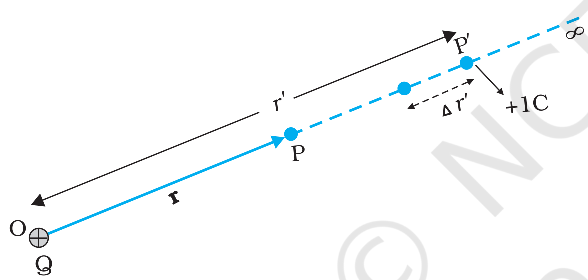

FIGURE 2.3 Work done in bringing a unit positive test charge from infinity to the point P, against the repulsive force of charge Q ($Q > 0$), is the potential at P due to the charge Q.

At some intermediate point P′ on the path, the electrostatic force on a unit positive charge is

$$\frac{Q \times 1}{4\pi\varepsilon_0 r'^2}\hat{\mathbf{r}}' \tag{2.5}$$

where $\hat{\mathbf{r}}'$ is the unit vector along OP′. Work done against this force from $\mathbf{r}'$ to $\mathbf{r}' + \Delta\mathbf{r}'$ is

$$\Delta W = -\frac{Q}{4\pi\varepsilon_0 r'^2}\Delta r' \tag{2.6}$$

The negative sign appears because for $\Delta r' < 0$, ΔW is positive . Total work done (W) by the external force is obtained by integrating Eq. (2.6) from $r' = \infty$ to $r' = r$,

$$W = -\int_\infty^r \frac{Q}{4\pi\varepsilon_0 r'^2}dr' = \left.\frac{Q}{4\pi\varepsilon_0 r'}\right|_\infty^r = \frac{Q}{4\pi\varepsilon_0 r} \tag{2.7}$$

This, by definition is the potential at P due to the charge Q

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r} \tag{2.8}$$

Equation (2.8) is true for any sign of the charge Q, though we considered $Q > 0$ in its derivation. For $Q < 0$, $V < 0$, i.e., work done (by the external force) per unit positive test charge in bringing it from infinity to the point is negative. This is equivalent to saying that work done by the electrostatic force in bringing the unit positive charge form infinity to the point P is positive. [This is as it should be, since for $Q < 0$, the force on a unit positive test charge is attractive, so that the electrostatic force and the displacement (from infinity to P) are in the same direction.] Finally, we note that Eq. (2.8) is consistent with the choice that potential at infinity be zero.

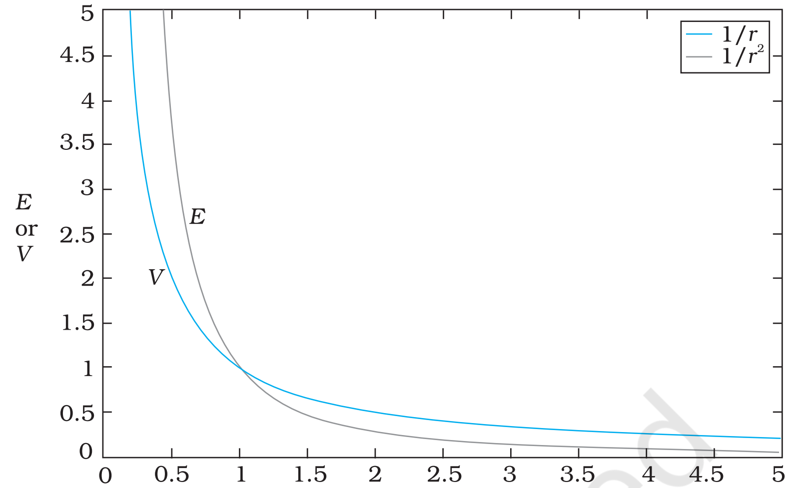

FIGURE 2.4 Variation of potential V with r [in units of $(Q/4\pi\varepsilon_0)$ m^{-1}] (blue curve) and field with r [in units of $(Q/4\pi\varepsilon_0)$ m^{-2}] (black curve) for a point charge Q.

Figure (2.4) shows how the electrostatic potential ($\propto 1/r$) and the electrostatic field ($\propto 1/r^2$) varies with r.

Example 2.1

(a) Calculate the potential at a point P due to a charge of 4×10^{-7}C located 9 cm away.

(b) Hence obtain the work done in bringing a charge of 2×10^{-9} C from infinity to the point P. Does the answer depend on the path along which the charge is brought?

Solution

(a) $V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r} = 9\times10^9 \text{ Nm}^2 \text{ C}^{-2} \times \dfrac{4\times10^{-7}\text{ C}}{0.09\text{ m}}$

$= 4 \times 10^4$ V

(b) $W = qV = 2 \times 10^{-9}\text{C} \times 4 \times 10^4\text{V}$

$= 8 \times 10^{-5}$ J

No, work done will be path independent. Any arbitrary infinitesimal path can be resolved into two perpendicular displacements: One along **r** and another perpendicular to **r**. The work done corresponding to the later will be zero.

EXAMPLE 2.1

2.4 Potential due to an Electric Dipole

As we learnt in the last chapter, an electric dipole consists of two charges q and $-q$ separated by a (small) distance $2a$. Its total charge is zero. It is characterised by a dipole moment vector **p** whose magnitude is $q \times 2a$ and which points in the direction from $-q$ to q (Fig. 2.5). We also saw that the electric field of a dipole at a point with position vector **r** depends not just on the magnitude r, but also on the angle between **r** and **p**. Further,

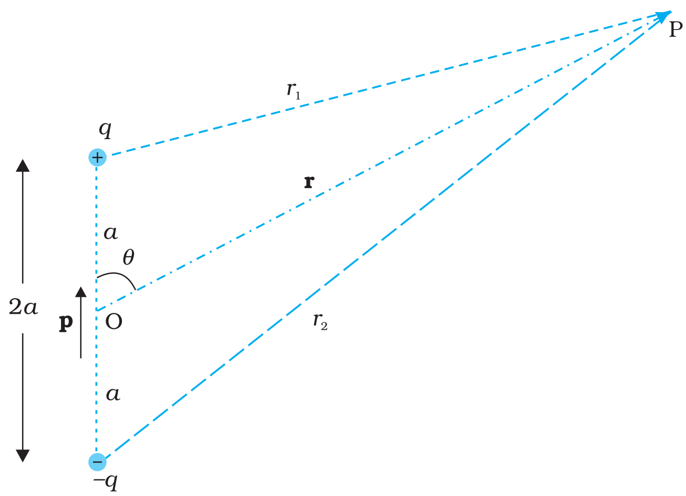

FIGURE 2.5 Quantities involved in the calculation of potential due to a dipole.

the field falls off, at large distance, not as $1/r^2$ (typical of field due to a single charge) but as $1/r^3$. We, now, determine the electric potential due to a dipole and contrast it with the potential due to a single charge.

As before, we take the origin at the centre of the dipole. Now we know that the electric field obeys the superposition principle. Since potential is related to the work done by the field, electrostatic potential also follows the superposition principle. Thus, the potential due to the dipole is the sum of potentials due to the charges q and $-q$

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right) \tag{2.9}$$

where r_1 and r_2 are the distances of the point P from q and $-q$, respectively.

Now, by geometry,

$$r_1^2 = r^2 + a^2 - 2ar\cos\theta$$

$$r_2^2 = r^2 + a^2 + 2ar\cos\theta \tag{2.10}$$

We take r much greater than a ($r >> a$) and retain terms only upto the first order in a/r

$$\cong r^2\left(1 - \frac{2a\cos\theta}{r}\right) \tag{2.11}$$

Similarly,

$$r_2^2 \cong r^2\left(1 + \frac{2a\cos\theta}{r}\right) \tag{2.12}$$

Using the Binomial theorem and retaining terms upto the first order in a/r; we obtain,

$$\frac{1}{r_1} \cong \frac{1}{r}\left(1 - \frac{2a\cos\theta}{r}\right)^{-1/2} \cong \frac{1}{r}\left(1 + \frac{a}{r}\cos\theta\right) \tag{2.13(a)}$$

$$\frac{1}{r_2} \cong \frac{1}{r}\left(1 + \frac{2a\cos\theta}{r}\right)^{-1/2} \cong \frac{1}{r}\left(1 - \frac{a}{r}\cos\theta\right) \tag{2.13(b)}$$

Using Eqs. (2.9) and (2.13) and $p = 2qa$, we get

$$V = \frac{q}{4\pi\varepsilon_0}\frac{2a\cos\theta}{r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} \tag{2.14}$$

Now, $p\cos\theta = \mathbf{p}\cdot\hat{\mathbf{r}}$

where $\hat{\mathbf{r}}$ is the unit vector along the position vector **OP**.

The electric potential of a dipole is then given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}; \qquad (r >> a) \tag{2.15}$$

Equation (2.15) is, as indicated, approximately true only for distances large compared to the size of the dipole, so that higher order terms in a/r are negligible. For a point dipole **p** at the origin, Eq. (2.15) is, however, exact.

From Eq. (2.15), potential on the dipole axis ($\theta = 0, \pi$) is given by

$$V = \pm\frac{1}{4\pi\varepsilon_0}\frac{p}{r^2} \tag{2.16}$$

(Positive sign for $\theta = 0$, negative sign for $\theta = \pi$.) The potential in the equatorial plane ($\theta = \pi/2$) is zero.

The important contrasting features of electric potential of a dipole from that due to a single charge are clear from Eqs. (2.8) and (2.15):

(i) The potential due to a dipole depends not just on r but also on the angle between the position vector **r** and the dipole moment vector **p**. (It is, however, axially symmetric about **p**. That is, if you rotate the position vector **r** about **p**, keeping θ fixed, the points corresponding to P on the cone so generated will have the same potential as at P.)

(ii) The electric dipole potential falls off, at large distance, as $1/r^2$, not as $1/r$, characteristic of the potential due to a single charge. (You can refer to the Fig. 2.5 for graphs of $1/r^2$ versus r and $1/r$ versus r, drawn there in another context.)

2.5 Potential due to a System of Charges

Consider a system of charges $q_1, q_2, \ldots, q_n$ with position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ relative to some origin (Fig. 2.6). The potential V_1 at P due to the charge q_1 is

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_{1P}}$$

where r_{1P} is the distance between q_1 and P.

Similarly, the potential V_2 at P due to q_2 and V_3 due to q_3 are given by

$$V_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_{2P}}, \quad V_3 = \frac{1}{4\pi\varepsilon_0}\frac{q_3}{r_{3P}}$$

where r_{2P} and r_{3P} are the distances of P from charges q_2 and q_3, respectively; and so on for the potential due to other charges. By the superposition principle, the potential V at P due to the total charge configuration is the algebraic sum of the potentials due to the individual charges

$$V = V_1 + V_2 + \ldots + V_n \tag{2.17}$$

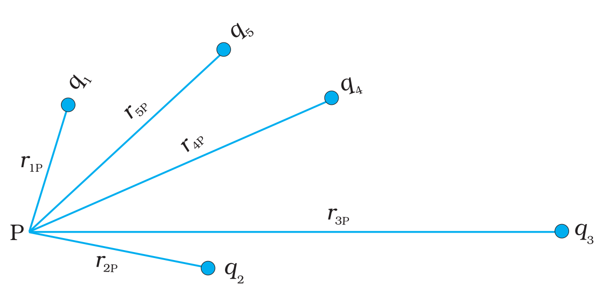

FIGURE 2.6 Potential at a point due to a system of charges is the sum of potentials due to individual charges.

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{1P}} + \frac{q_2}{r_{2P}} + \ldots\ldots + \frac{q_n}{r_{nP}}\right) \qquad (2.18)$$

If we have a continuous charge distribution characterised by a charge density $\rho(\mathbf{r})$, we divide it, as before, into small volume elements each of size Δv and carrying a charge $\rho\Delta v$. We then determine the potential due to each volume element and sum (strictly speaking , integrate) over all such contributions, and thus determine the potential due to the entire distribution.

We have seen in Chapter 1 that for a uniformly charged spherical shell, the electric field outside the shell is as if the entire charge is concentrated at the centre. Thus, the potential outside the shell is given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \quad (r \geq R) \qquad [2.19(a)]$$

where q is the total charge on the shell and R its radius. The electric field inside the shell is zero. This implies (Section 2.6) that potential is constant inside the shell (as no work is done in moving a charge inside the shell), and, therefore, equals its value at the surface, which is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \qquad [2.19(b)]$$

Example 2.2 Two charges 3×10^{-8} C and -2×10^{-8} C are located 15 cm apart. At what point on the line joining the two charges is the electric potential zero? Take the potential at infinity to be zero.

Solution Let us take the origin O at the location of the positive charge. The line joining the two charges is taken to be the *x*-axis; the negative charge is taken to be on the right side of the origin (Fig. 2.7).

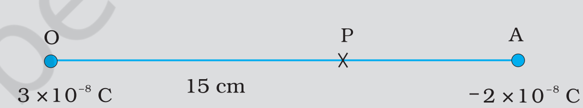

FIGURE 2.7

Let P be the required point on the *x*-axis where the potential is zero. If x is the *x*-coordinate of P, obviously x must be positive. (There is no possibility of potentials due to the two charges adding up to zero for $x < 0$.) If x lies between O and A, we have

$$\frac{1}{4\pi\varepsilon_0}\left[\frac{3\times10^{-8}}{x\times10^{-2}} - \frac{2\times10^{-8}}{(15-x)\times10^{-2}}\right] = 0$$

where x is in cm. That is,

$$\frac{3}{x} - \frac{2}{15-x} = 0$$

which gives x = 9 cm.

If x lies on the extended line OA, the required condition is

$$\frac{3}{x} - \frac{2}{x-15} = 0$$

EXAMPLE 2.2

which gives

$x = 45$ cm

Thus, electric potential is zero at 9 cm and 45 cm away from the positive charge on the side of the negative charge. Note that the formula for potential used in the calculation required choosing potential to be zero at infinity.

EXAMPLE 2.2

Example 2.3 Figures 2.8 (a) and (b) show the field lines of a positive and negative point charge respectively.

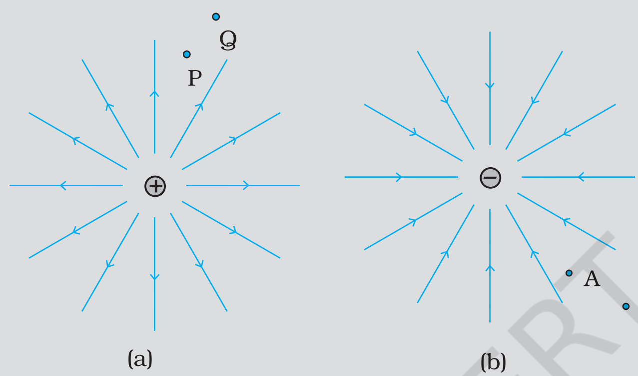

FIGURE 2.8

(a) Give the signs of the potential difference $V_P - V_Q$; $V_B - V_A$.

(b) Give the sign of the potential energy difference of a small negative charge between the points Q and P; A and B.

(c) Give the sign of the work done by the field in moving a small positive charge from Q to P.

(d) Give the sign of the work done by the external agency in moving a small negative charge from B to A.

(e) Does the kinetic energy of a small negative charge increase or decrease in going from B to A?

Solution

(a) As $V \propto \frac{1}{r}$, $V_P > V_Q$. Thus, $(V_P - V_Q)$ is positive. Also V_B is less negative than V_A. Thus, $V_B > V_A$ or $(V_B - V_A)$ is positive.

(b) A small negative charge will be attracted towards positive charge. The negative charge moves from higher potential energy to lower potential energy. Therefore the sign of potential energy difference of a small negative charge between Q and P is positive. Similarly, $(P.E.)_A > (P.E.)_B$ and hence sign of potential energy differences is positive.

(c) In moving a small positive charge from Q to P, work has to be done by an external agency against the electric field. Therefore, work done by the field is negative.

(d) In moving a small negative charge from B to A work has to be done by the external agency. It is positive.

(e) Due to force of repulsion on the negative charge, velocity decreases and hence the kinetic energy decreases in going from B to A.

EXAMPLE 2.3

PHYSICS

Electric potential, equipotential surfaces:
http://video.mit.edu/watch/4-electrostatic-potential-elctric-energy-ev-conservative-field-equipotential-sufaces-12584/

2.6 Equipotential Surfaces

An equipotential surface is a surface with a constant value of potential at all points on the surface. For a single charge q, the potential is given by Eq. (2.8):

$$V = \frac{1}{4\pi\varepsilon_o}\frac{q}{r}$$

This shows that V is a constant if r is constant . Thus, equipotential surfaces of a single point charge are concentric spherical surfaces centred at the charge.

Now the electric field lines for a single charge q are radial lines starting from or ending at the charge, depending on whether q is positive or negative. Clearly, the electric field at every point is normal to the equipotential surface passing through that point. This is true in general: *for any charge configuration, equipotential surface through a point is normal to the electric field at that point*. The proof of this statement is simple.

If the field were not normal to the equipotential surface, it would have non-zero component along the surface. To move a unit test charge against the direction of the component of the field, work would have to be done. But this is in contradiction to the definition of an equipotential surface: there is no potential difference between any two points on the surface and no work is required to move a test charge on the surface. The electric field must, therefore, be normal to the equipotential surface at every point. Equipotential surfaces offer an alternative visual picture in addition to the picture of electric field lines around a charge configuration.

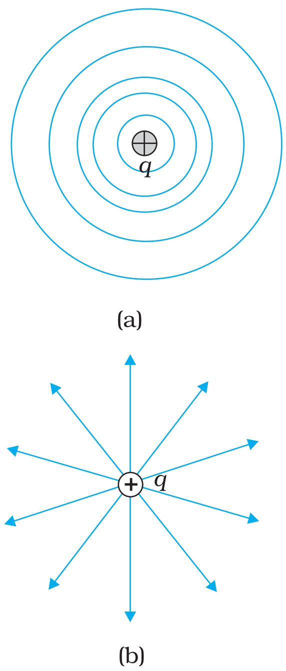

FIGURE 2.9 For a single charge q (a) equipotential surfaces are spherical surfaces centred at the charge, and (b) electric field lines are radial, starting from the charge if $q > 0$.

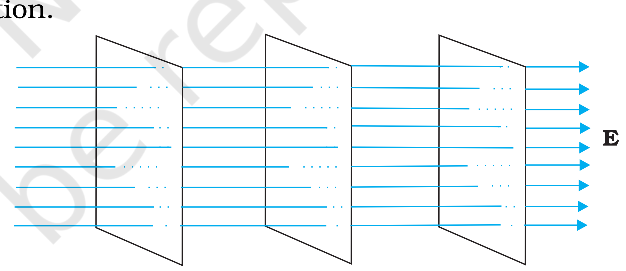

FIGURE 2.10 Equipotential surfaces for a uniform electric field.

For a uniform electric field **E**, say, along the x -axis, the equipotential surfaces are planes normal to the x -axis, i.e., planes parallel to the y-z plane (Fig. 2.10). Equipotential surfaces for (a) a dipole and (b) two identical positive charges are shown in Fig. 2.11.

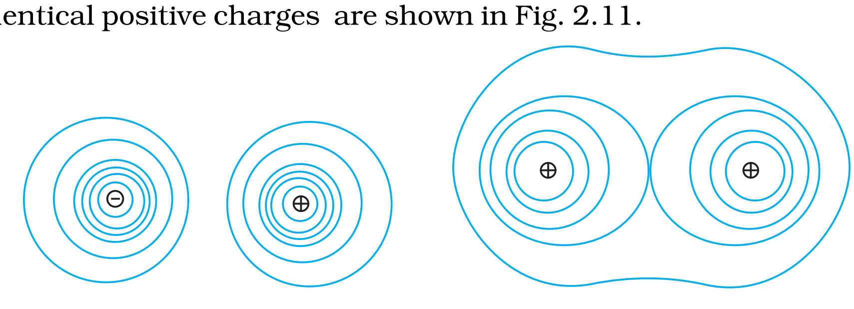

FIGURE 2.11 Some equipotential surfaces for (a) a dipole, (b) two identical positive charges.

2.6.1 Relation between field and potential

Consider two closely spaced equipotential surfaces A and B (Fig. 2.12) with potential values V and $V + \delta V$, where δV is the change in V in the direction of the electric field **E**. Let P be a point on the surface B. δl is the perpendicular distance of the surface A from P. Imagine that a unit positive charge is moved along this perpendicular from the surface B to surface A against the electric field. The work done in this process is $|\mathbf{E}|\,\delta l$.

This work equals the potential difference $V_A - V_B$.

Thus,

$$|\mathbf{E}|\,\delta l = V - (V + \delta V) = -\delta V$$

i.e., $$|\mathbf{E}| = -\frac{\delta V}{\delta l} \qquad (2.20)$$

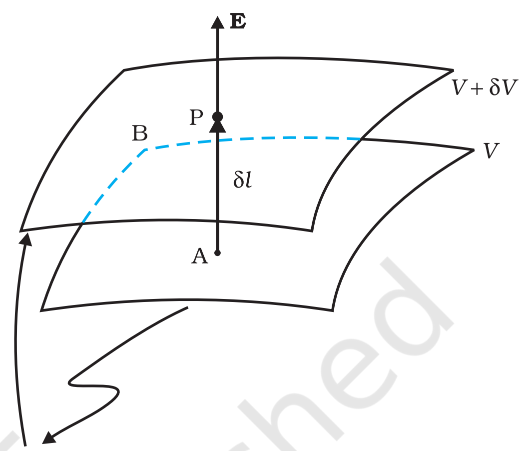

FIGURE 2.12 From the potential to the field.

Since δV is negative, $\delta V = -|\delta V|$. we can rewrite Eq (2.20) as

$$|\mathbf{E}| = -\frac{\delta V}{\delta l} = +\frac{|\delta V|}{\delta l} \qquad (2.21)$$

We thus arrive at two important conclusions concerning the relation between electric field and potential:

(i) *Electric field is in the direction in which the potential decreases steepest.*

(ii) *Its magnitude is given by the change in the magnitude of potential per unit displacement normal to the equipotential surface at the point.*

2.7 Potential Energy of a System of Charges

Consider first the simple case of two charges q_1 and q_2 with position vector $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to some origin. Let us calculate the work done (externally) in building up this configuration. This means that we consider the charges q_1 and q_2 initially at infinity and determine the work done by an external agency to bring the charges to the given locations. Suppose, first the charge q_1 is brought from infinity to the point $\mathbf{r}_1$. There is no external field against which work needs to be done, so work done in bringing q_1 from infinity to $\mathbf{r}_1$ is zero. This charge produces a potential in space given by

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_{1P}}$$

where r_{1P} is the distance of a point P in space from the location of q_1. From the definition of potential, work done in bringing charge q_2 from infinity to the point $\mathbf{r}_2$ is q_2 times the potential at $\mathbf{r}_2$ due to q_1:

$$\text{work done on } q_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

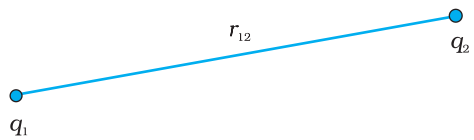

FIGURE 2.13 Potential energy of a system of charges q_1 and q_2 is directly proportional to the product of charges and inversely to the distance between them.

where r_{12} is the distance between points 1 and 2.

Since electrostatic force is conservative, this work gets stored in the form of potential energy of the system. Thus, the potential energy of a system of two charges q_1 and q_2 is

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}} \qquad (2.22)$$

Obviously, if q_2 was brought first to its present location and q_1 brought later, the potential energy U would be the same. More generally, the potential energy expression, Eq. (2.22), is unaltered whatever way the charges are brought to the specified locations, because of path-independence of work for electrostatic force.

Equation (2.22) is true for any sign of q_1 and q_2. If $q_1 q_2 > 0$, potential energy is positive. This is as expected, since for like charges ($q_1 q_2 > 0$), electrostatic force is repulsive and a positive amount of work is needed to be done against this force to bring the charges from infinity to a finite distance apart. For unlike charges ($q_1\, q_2 < 0$), the electrostatic force is attractive. In that case, a positive amount of work is needed against this force to take the charges from the given location to infinity. In other words, a negative amount of work is needed for the reverse path (from infinity to the present locations), so the potential energy is negative.

Equation (2.22) is easily generalised for a system of any number of point charges. Let us calculate the potential energy of a system of three charges q_1, q_2 and q_3 located at $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, respectively. To bring q_1 first from infinity to $\mathbf{r}_1$, no work is required. Next we bring q_2 from infinity to $\mathbf{r}_2$. As before, work done in this step is

$$q_2 V_1(\mathbf{r}_2) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}} \qquad (2.23)$$

The charges q_1 and q_2 produce a potential, which at any point P is given by

$$V_{1,2} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{1P}} + \frac{q_2}{r_{2P}}\right) \qquad (2.24)$$

Work done next in bringing q_3 from infinity to the point $\mathbf{r}_3$ is q_3 times $V_{1,\,2}$ at $\mathbf{r}_3$

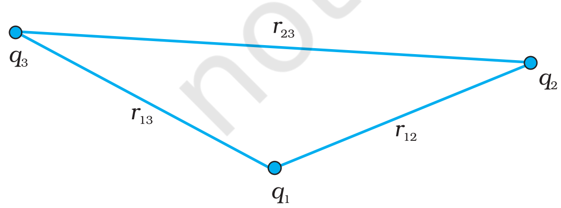

FIGURE 2.14 Potential energy of a system of three charges is given by Eq. (2.26), with the notation given in the figure.

$$q_3 V_{1,2}(\mathbf{r}_3) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right) \qquad (2.25)$$

The total work done in assembling the charges at the given locations is obtained by adding the work done in different steps [Eq. (2.23) and Eq. (2.25)],

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right) \qquad (2.26)$$

Again, because of the conservative nature of the electrostatic force (or equivalently, the path independence of work done), the final expression for U, Eq. (2.26), is independent of the manner in which the configuration is assembled. *The potential energy* *is characteristic of the present state of configuration, and not the way the state is achieved.*

Example 2.4 Four charges are arranged at the corners of a square ABCD of side *d*, as shown in Fig. 2.15.(a) Find the work required to put together this arrangement. (b) A charge q_0 is brought to the centre E of the square, the four charges being held fixed at its corners. How much extra work is needed to do this?

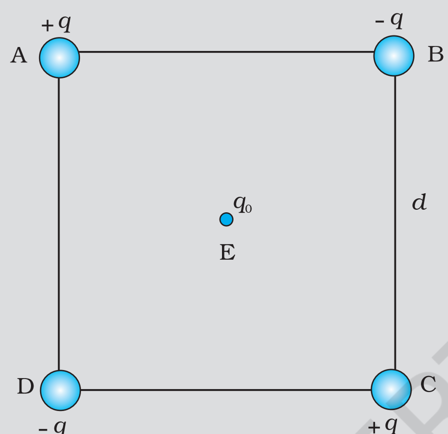

FIGURE 2.15

Solution

(a) Since the work done depends on the final arrangement of the charges, and not on how they are put together, we calculate work needed for one way of putting the charges at A, B, C and D. Suppose, first the charge $+q$ is brought to A, and then the charges $-q$, $+q$, and $-q$ are brought to B, C and D, respectively. The total work needed can be calculated in steps:

(i) Work needed to bring charge $+q$ to A when no charge is present elsewhere: this is zero.

(ii) Work needed to bring $-q$ to B when $+q$ is at A. This is given by (charge at B) × (electrostatic potential at B due to charge $+q$ at A)

$$= -q \times \left(\frac{q}{4\pi\varepsilon_0 d}\right) = -\frac{q^2}{4\pi\varepsilon_0 d}$$

(iii) Work needed to bring charge $+q$ to C when $+q$ is at A and $-q$ is at B. This is given by (charge at C) × (potential at C due to charges at A and B)

$$= +q\left(\frac{+q}{4\pi\varepsilon_0 d\sqrt{2}} + \frac{-q}{4\pi\varepsilon_0 d}\right)$$

$$= \frac{-q^2}{4\pi\varepsilon_0 d}\left(1 - \frac{1}{\sqrt{2}}\right)$$

(iv) Work needed to bring $-q$ to D when $+q$ at A,$-q$ at B, and $+q$ at C. This is given by (charge at D) × (potential at D due to charges at A, B and C)

$$= -q\left(\frac{+q}{4\pi\varepsilon_0 d} + \frac{-q}{4\pi\varepsilon_0 d\sqrt{2}} + \frac{q}{4\pi\varepsilon_0 d}\right)$$

$$= \frac{-q^2}{4\pi\varepsilon_0 d}\left(2 - \frac{1}{\sqrt{2}}\right)$$

EXAMPLE 2.4

Add the work done in steps (i), (ii), (iii) and (iv). The total work required is

$$= \frac{-q^2}{4\pi\varepsilon_0 d}\left\{(0)+(1)+\left(1-\frac{1}{\sqrt{2}}\right)+\left(2-\frac{1}{\sqrt{2}}\right)\right\}$$

$$= \frac{-q^2}{4\pi\varepsilon_0 d}\left(4-\sqrt{2}\right)$$

The work done depends only on the arrangement of the charges, and not how they are assembled. By definition, this is the total electrostatic energy of the charges.

(Students may try calculating same work/energy by taking charges in any other order they desire and convince themselves that the energy will remain the same.)

(b) The extra work necessary to bring a charge q_0 to the point E when the four charges are at A, B, C and D is $q_0 \times$ (electrostatic potential at E due to the charges at A, B, C and D). The electrostatic potential at E is clearly zero since potential due to A and C is cancelled by that due to B and D. Hence no work is required to bring any charge to point E.

EXAMPLE 2.4

2.8 Potential Energy in an External Field

2.8.1 Potential energy of a single charge

In Section 2.7, the source of the electric field was specified – the charges and their locations - and the potential energy of the system of those charges was determined. In this section, we ask a related but a distinct question. What is the potential energy of a charge q in a given field? This question was, in fact, the starting point that led us to the notion of the electrostatic potential (Sections 2.1 and 2.2). But here we address this question again to clarify in what way it is different from the discussion in Section 2.7.

The main difference is that we are now concerned with the potential energy of a charge (or charges) in an *external* field. The external field **E** is *not* produced by the given charge(s) whose potential energy we wish to calculate. **E** is produced by sources external to the given charge(s).The external sources may be known, but often they are unknown or unspecified; what is specified is the electric field **E** or the electrostatic potential V due to the external sources. We assume that the charge q does not significantly affect the sources producing the external field. This is true if q is very small, or the external sources are held fixed by other unspecified forces. Even if q is finite, its influence on the external sources may still be ignored in the situation when very strong sources far away at infinity produce a finite field **E** in the region of interest. Note again that we are interested in determining the potential energy of a given charge q (and later, a system of charges) in the external field; we are not interested in the potential energy of the sources producing the external electric field.

The external electric field **E** and the corresponding external potential V may vary from point to point. By definition, V at a point P is the work done in bringing a unit positive charge from infinity to the point P.

(We continue to take potential at infinity to be zero.) Thus, work done in bringing a charge q from infinity to the point P in the external field is qV. This work is stored in the form of potential energy of q. If the point P has position vector **r** relative to some origin, we can write:

Potential energy of q at **r** in an external field

$$= qV(\mathbf{r}) \tag{2.27}$$

where V(**r**) is the external potential at the point **r**.

Thus, if an electron with charge $q = e = 1.6\times10^{-19}$ C is accelerated by a potential difference of $\Delta V = 1$ volt, it would gain energy of $q\Delta V = 1.6 \times 10^{-19}$J. This unit of energy is defined as 1 *electron volt* or 1eV, i.e., 1 eV=1.6×10^{-19}J. The units based on eV are most commonly used in atomic, nuclear and particle physics, (1 keV = 10^3eV = 1.6×10^{-16}J, 1 MeV = 10^6eV = 1.6×10^{-13}J, 1 GeV = 10^9eV = 1.6×10^{-10}J and 1 TeV = 10^{12}eV = 1.6×10^{-7}J). [This has already been defined on Page 117, XI Physics Part I, Table 6.1.]

2.8.2 Potential energy of a system of two charges in an external field

Next, we ask: what is the potential energy of a system of two charges q_1 and q_2 located at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, in an external field? First, we calculate the work done in bringing the charge q_1 from infinity to $\mathbf{r}_1$. Work done in this step is $q_1\ V(\mathbf{r}_1)$, using Eq. (2.27). Next, we consider the work done in bringing q_2 to $\mathbf{r}_2$. In this step, work is done not only against the external field **E** but also against the field due to q_1.

Work done on q_2 against the external field

$= q_2\ V(\mathbf{r}_2)$

Work done on q_2 against the field due to q_1

$$= \frac{q_1 q_2}{4\pi\varepsilon_o r_{12}}$$

where r_{12} is the distance between q_1 and q_2. We have made use of Eqs. (2.27) and (2.22). By the superposition principle for fields, we add up the work done on q_2 against the two fields (**E** and that due to q_1):

Work done in bringing q_2 to $\mathbf{r}_2$

$$= q_2 V(\mathbf{r}_2) + \frac{q_1 q_2}{4\pi\varepsilon_o r_{12}} \tag{2.28}$$

Thus,

Potential energy of the system

= the total work done in assembling the configuration

$$= q_1 V(\mathbf{r}_1) + q_2 V(\mathbf{r}_2) + \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}} \tag{2.29}$$

Example 2.5

(a) Determine the electrostatic potential energy of a system consisting of two charges 7 μC and –2 μC (and with no external field) placed at (–9 cm, 0, 0) and (9 cm, 0, 0) respectively.

(b) How much work is required to separate the two charges infinitely away from each other?

EXAMPLE 2.5

Example 2.5

(c) Suppose that the same system of charges is now placed in an external electric field $E = A\,(1/r^2)$; $A = 9 \times 10^5$ C m^{-2}. What would the electrostatic energy of the configuration be?

Solution

(a) $U = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1 q_2}{r} = 9\times10^9 \times \dfrac{7\times(-2)\times10^{-12}}{0.18} = -0.7$ J.

(b) $W = U_2 - U_1 = 0 - U = 0 - (-0.7) = 0.7$ J.

(c) The mutual interaction energy of the two charges remains unchanged. In addition, there is the energy of interaction of the two charges with the external electric field. We find,

$$q_1 V(\mathbf{r}_1) + q_2 V(\mathbf{r}_2) = A\frac{7\mu\text{C}}{0.09\text{m}} + A\frac{-2\mu\text{C}}{0.09\text{m}}$$

and the net electrostatic energy is

$$q_1 V(\mathbf{r}_1) + q_2 V(\mathbf{r}_2) + \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}} = A\frac{7\ \mu\text{C}}{0.09\,\text{m}} + A\frac{-2\ \mu\text{C}}{0.09\,\text{m}} - 0.7\,\text{J}$$

$$= 70 - 20 - 0.7 = 49.3\ \text{J}$$

2.8.3 Potential energy of a dipole in an external field

Consider a dipole with charges $q_1 = +q$ and $q_2 = -q$ placed in a uniform electric field **E**, as shown in Fig. 2.16.

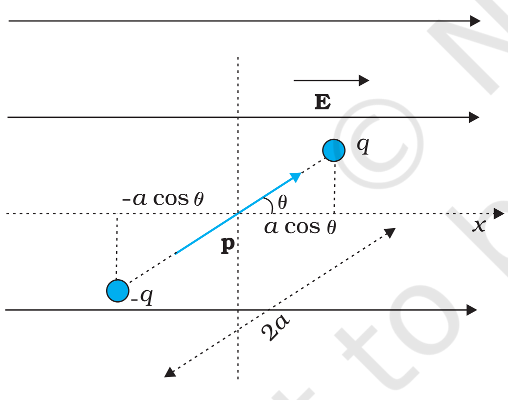

FIGURE 2.16 Potential energy of a dipole in a uniform external field.

As seen in the last chapter, in a uniform electric field, the dipole experiences no net force; but experiences a torque **τ** given by

$$\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E} \qquad (2.30)$$

which will tend to rotate it (unless **p** is parallel or antiparallel to **E**). Suppose an external torque $\boldsymbol{\tau}_{ext}$ is applied in such a manner that it just neutralises this torque and rotates it in the plane of paper from angle θ_0 to angle θ_1 at an infinitesimal angular speed and *without angular acceleration*. The amount of work done by the external torque will be given by

$$W = \int_{\theta_0}^{\theta_1} \tau_{ext}(\theta)\,d\theta = \int_{\theta_0}^{\theta_1} pE\sin\theta\,d\theta$$

$$= pE\left(\cos\theta_0 - \cos\theta_1\right) \qquad (2.31)$$

This work is stored as the potential energy of the system. We can then associate potential energy $U(\theta)$ with an inclination θ of the dipole. Similar to other potential energies, there is a freedom in choosing the angle where the potential energy U is taken to be zero. A natural choice is to take $\theta_0 = \pi/2$. (An explanation for it is provided towards the end of discussion.) We can then write,

$$U(\theta) = pE\left(\cos\frac{\pi}{2} - \cos\theta\right) = -pE\cos\theta = -\mathbf{p}\cdot\mathbf{E} \qquad (2.32)$$

This expression can alternately be understood also from Eq. (2.29). We apply Eq. (2.29) to the present system of two charges $+q$ and $–q$. The potential energy expression then reads

$$U'(\theta) = q[V(\mathbf{r}_1) - V(\mathbf{r}_2)] - \frac{q^2}{4\pi\varepsilon_0 \times 2a} \qquad (2.33)$$

Here, $\mathbf{r_1}$ and $\mathbf{r_2}$ denote the position vectors of $+q$ and $–q$. Now, the potential difference between positions $\mathbf{r_1}$ and $\mathbf{r_2}$ equals the work done in bringing a unit positive charge against field from $\mathbf{r_2}$ to $\mathbf{r_1}$. The displacement parallel to the force is $2a\cos\theta$. Thus, $[V(\mathbf{r}_1) - V(\mathbf{r}_2)] = -E \times 2a\cos\theta$. We thus obtain,

$$U'(\theta) = -pE\cos\theta - \frac{q^2}{4\pi\varepsilon_0 \times 2a} = -\mathbf{p}\cdot\mathbf{E} - \frac{q^2}{4\pi\varepsilon_0 \times 2a} \qquad (2.34)$$

We note that $U'(\theta)$ differs from $U(\theta)$ by a quantity which is just a constant for a given dipole. Since a constant is insignificant for potential energy, we can drop the second term in Eq. (2.34) and it then reduces to Eq. (2.32).

We can now understand why we took $\theta_0=\pi/2$. In this case, the work done against the *external* field **E** in bringing $+q$ and $-q$ are equal and opposite and cancel out, i.e., $q[V(\mathbf{r}_1) - V(\mathbf{r}_2)]=0$.

Example 2.6 A molecule of a substance has a permanent electric dipole moment of magnitude 10^{-29} C m. A mole of this substance is polarised (at low temperature) by applying a strong electrostatic field of magnitude 10^6 V m^{-1}. The direction of the field is suddenly changed by an angle of 60°. Estimate the heat released by the substance in aligning its dipoles along the new direction of the field. For simplicity, assume 100% polarisation of the sample.

Solution Here, dipole moment of each molecules = 10^{-29} C m
As 1 mole of the substance contains 6×10^{23} molecules,
total dipole moment of all the molecules, $p = 6 \times 10^{23} \times 10^{-29}$ C m
$= 6 \times 10^{-6}$ C m

Initial potential energy, $U_i = -pE\cos\theta = -6\times10^{-6}\times10^{6}\cos 0° = -6$ J
Final potential energy (when $\theta = 60°$), $U_f = -6 \times 10^{-6} \times 10^{6}\cos 60° = -3$ J
Change in potential energy = –3 J – (–6J) = 3 J
So, there is loss in potential energy. This must be the energy released by the substance in the form of heat in aligning its dipoles.

EXAMPLE 2.6

2.9 Electrostatics of Conductors

Conductors and insulators were described briefly in Chapter 1. Conductors contain mobile charge carriers. In metallic conductors, these charge carriers are electrons. In a metal, the outer (valence) electrons part away from their atoms and are free to move. These electrons are free within the metal but not free to leave the metal. The free electrons form a kind of 'gas'; they collide with each other and with the ions, and move randomly in different directions. In an external electric field, they drift against the direction of the field. The positive ions made up of the nuclei and the bound electrons remain held in their fixed positions. In electrolytic conductors, the charge carriers are both positive and negative ions; but