PYQ1

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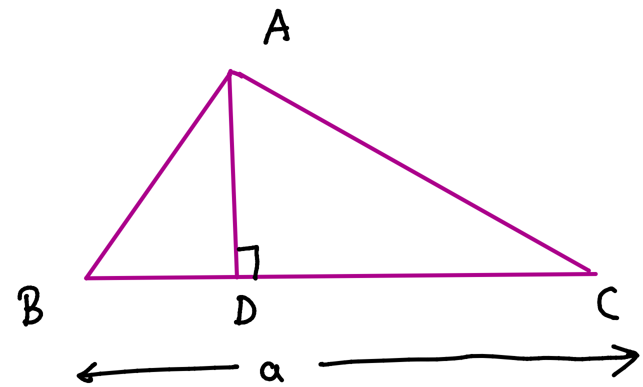
Question. In a triangle of base a, the ratio of the other sides is $r\,(<1)$. Show that the altitudes of the triangle is less than or equal to $\dfrac{ar}{1-r^2}$. (Also prove that when the altitude is greatest, the vertical angle A of the triangle is $\dfrac{\pi}{2} - 2\tan^{-1} r$.

[Subjective, IIT JEE 1991, 4]

Solution.



Let $\dfrac{AB}{AC} = r\,(<1)$.

$BC = a$

& h be the altitude AD.

$\therefore$ By Sine law in ΔABC,

$$\frac{AB}{\sin C} = \frac{AC}{\sin B} = \frac{a}{\sin A} \qquad \text{—— (1)}$$

From ΔADB, $h = AB \sin B$

$$= a\left(\frac{\sin C}{\sin A}\right)\sin B \qquad \text{[Using equation (1)]}$$

$$= \frac{a \sin B \cdot \sin C}{\sin[\pi - (B+C)]}$$

$$= a \sin B \cdot \sin C$$

$$= \frac{a \sin B . \sin C}{\sin(B+C)}$$

$$= \frac{a \sin B . \sin C . \sin(B-C)}{\sin(B+C) . \sin(B-C)}$$

$$= \frac{a \sin B . \sin C . \sin(B-C)}{\sin^2 B - \sin^2 C}$$

$$= a \frac{\sin B \sin C}{\sin^2 B . \left\{1 - \frac{\sin^2 C}{\sin^2 B}\right\}} . \sin(B-C)$$

$$= \frac{a . \frac{\sin C}{\sin B}}{1 - \frac{\sin^2 C}{\sin^2 B}} . \sin(B-C) \quad \text{—— (2)}$$

Now, we know that $\frac{AB}{AC} = r$

$\Rightarrow$ Using Sine law, $\frac{\sin C}{\sin B} = r$ —— (3)

$\therefore$ From equation 2,

$$R = \frac{ar}{1-r^2} . \sin(B-C)$$

$\because \sin(B-C) \leq 1$

$$\Rightarrow \boxed{h \leq \frac{ar}{1-r^2}}$$ Proved

Now, for h to be maximum,

$\sin(B-C) = 1$

$$\Rightarrow B - C = \frac{\pi}{2}$$

$$\Rightarrow B = C + \pi/2$$

$$\therefore r = \frac{\sin C}{\sin B} = \frac{\sin C}{\sin(C+\frac{\pi}{2})}$$

$$= \tan C$$

$$\Rightarrow C = \tan^{-1} r \quad \text{——— (4)}$$

Now, $A + B + C = \pi$

$$\Rightarrow A + C + \frac{\pi}{2} + C = \pi$$

$$\Rightarrow A = \frac{\pi}{2} - 2C$$

$$\Rightarrow \boxed{A = \frac{\pi}{2} - 2\tan^{-1} r}$$ Proved