Rolle's Theorem

Let $f : [a, b] \to R$ be a function satisfying the following:

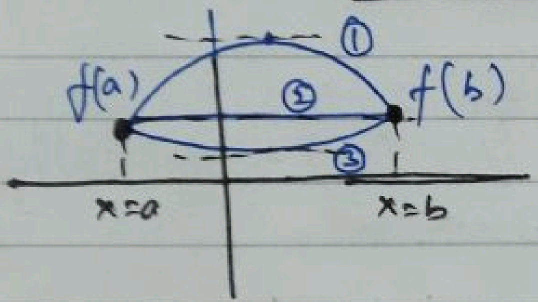
i) f is continous on closed interval $[a, b]$

ii) f is diff. on (a, b)

iii) $f(a) = f(b)$

Then, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

3 Cases Possible



In all three cases, $\exists c$, for which $f'(c) = 0$.

Q) Show that $f'(x) = 0$ have at least one solution for $f(x) = e^{\sin x} - \cos x$ in $x \in [0, 2\pi]$

$$f(0) = e^{\sin 0} - \cos 0 = e^0 - 1 = 1 - 1 = 0$$
$$f(2\pi) = e^{\sin 2\pi} - \cos 2\pi = 1 - 1 = 0$$

$$f(0) = f(2\pi)$$

$f(x)$ is cont. on $[0, 2\pi]$ and diff on $(0, 2\pi)$

Hence, by Rolle's Theorem, $\exists\ c \in (0, 2\pi)$ where $f'(c) = 0$

Mean Value Theorem (MVT / LMVT)

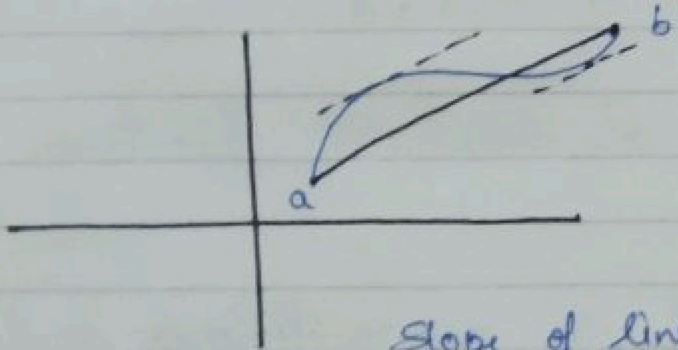
Let $f : [a, b] \to R$ be such that

i) $f(x)$ is cont. on closed interval $[a, b]$

ii) $f(x)$ is diff on open interval (a, b)

$\exists c \in (a, b)$ such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$



Slope of line joining $(a, f(a))$ & $(b, f(b))$ is $\frac{f(b) - f(a)}{b - a}$

Q) Prove that $|\sin x - \sin y| \leq |x - y|$

Let $f(t) = \sin t$

By MVT in interval (x, y)
$\exists c \in (x, y)$

$$f'(c) = \frac{\sin y - \sin x}{y - x}$$

$$\sin c = \frac{\sin y - \sin x}{y - x}$$

$$|\sin c| \leq 1$$

$$\left|\frac{\sin x - \sin y}{x - y}\right| \leq 1$$

$$|\sin x - \sin y| \leq |x - y|$$

→ Application of MVT

Collary I: Suppose $f : [a, b] \to R$ be a cont. and $f'(x) = 0 \;\forall\; x \in (a, b)$

Then, f must be a constant function.

Q) $f : (-\infty, \infty) \to (-\infty, \infty)$ is a continous and differentiable function. $f(2x) = f(x)$, $f(2) = 4$, find $f'(6) = ?$

A/Q $\quad f(2x) = f(x)$

we can say,

$$f(x) = f\left(\frac{x}{2}\right) = f\left(\frac{x}{4}\right) \text{------} = \lim_{n \to \infty} f\left(\frac{x}{2^n}\right)$$

So, $f(x) = \lim_{n \to \infty} f\left(\frac{x}{2^n}\right)$

$f(x) = f(0) \Rightarrow f(x)$ is a constant function.

So, $f'(x) = 0 \quad \forall \; x \in R$.

Cauchy Mean Value Theorem

Let f & g be two continous function on $[a, b]$ and differentiable on (a, b)

Then $\exists\, c \in (a, b)$ such that

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}}$$

Proof-

Let $h(x) = f(x) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g(x)$

so, $h(a) = h(b)$

By Rolle's theorem,

in (a, b) $\exists\, c$, such that.

$$h'(c) = 0$$

$$f'(c) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g'(c) = 0$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proved