

seen that:

$$\sin x = 0 \text{ gives } x = n\pi, \text{ where } n \in \mathbf{Z}$$

$$\cos x = 0 \text{ gives } x = (2n + 1)\frac{\pi}{2}, \text{ where } n \in \mathbf{Z}.$$

We shall now prove the following results:

Theorem 1 For any real numbers x and y ,

$$\sin x = \sin y \text{ implies } x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}$$

Proof If $\sin x = \sin y$, then

$$\sin x - \sin y = 0 \text{ or } 2\cos \frac{x+y}{2} \sin \frac{x-y}{2} = 0$$

which gives $\cos \frac{x+y}{2} = 0$ or $\sin \frac{x-y}{2} = 0$

Therefore $\frac{x+y}{2} = (2n+1)\frac{\pi}{2}$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

i.e. $x = (2n+1)\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

Hence $x = (2n+1)\pi + (-1)^{2n+1}y$ or $x = 2n\pi + (-1)^{2n}y$, where $n \in \mathbf{Z}$.

Combining these two results, we get

$$x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}.$$

Theorem 2 For any real numbers x and y , $\cos x = \cos y$, implies $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Proof If $\cos x = \cos y$, then

$$\cos x - \cos y = 0 \text{ i.e., } -2\sin \frac{x+y}{2} \sin \frac{x-y}{2} = 0$$

Thus $\sin \frac{x+y}{2} = 0$ or $\sin \frac{x-y}{2} = 0$

Therefore $\frac{x+y}{2} = n\pi$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

i.e. $x = 2n\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

Hence $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Theorem 3 Prove that if x and y are not odd multiple of $\frac{\pi}{2}$, then

$$\tan x = \tan y \text{ implies } x = n\pi + y, \text{ where } n \in \mathbf{Z}$$

Proof If $\tan x = \tan y$, then $\tan x - \tan y = 0$

or
$$\frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y} = 0$$

which gives $\sin(x - y) = 0$ (Why?)


Therefore $x - y = n\pi$, i.e., $x = n\pi + y$, where $n \in \mathbf{Z}$

Example 20 Find the solution of $\sin x = -\frac{\sqrt{3}}{2}$.

Solution We have $\sin x = -\frac{\sqrt{3}}{2} = -\sin \frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin \frac{4\pi}{3}$

Hence $\sin x = \sin \frac{4\pi}{3}$, which gives

$$x = n\pi + (-1)^n \frac{4\pi}{3}, \text{ where } n \in \mathbf{Z}.$$

 **Note** $\frac{4\pi}{3}$ is one such value of x for which $\sin x = -\frac{\sqrt{3}}{2}$. One may take any other value of x for which $\sin x = -\frac{\sqrt{3}}{2}$. The solutions obtained will be the same although these may apparently look different.

Example 21 Solve $\cos x = \frac{1}{2}$.

Solution We have, $\cos x = \frac{1}{2} = \cos \frac{\pi}{3}$

Therefore $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in \mathbf{Z}$.

Example 22 Solve $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$.

Solution We have, $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{2} + x + \frac{\pi}{3}\right)$

or
$$\tan 2x = \tan \left(x + \frac{5\pi}{6} \right)$$

Therefore
$$2x = n\pi + x + \frac{5\pi}{6}, \text{ where } n \in \mathbf{Z}$$

or
$$x = n\pi + \frac{5\pi}{6}, \text{ where } n \in \mathbf{Z}.$$

Example 23 Solve $\sin 2x - \sin 4x + \sin 6x = 0$.

Solution The equation can be written as

$$\sin 6x + \sin 2x - \sin 4x = 0$$

or
$$2 \sin 4x \cos 2x - \sin 4x = 0$$

i.e.
$$\sin 4x(2 \cos 2x - 1) = 0$$

Therefore
$$\sin 4x = 0 \quad \text{or} \quad \cos 2x = \frac{1}{2}$$

i.e.
$$\sin 4x = 0 \quad \text{or} \quad \cos 2x = \cos \frac{\pi}{3}$$

Hence
$$4x = n\pi \quad \text{or} \quad 2x = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in \mathbf{Z}$$

i.e.
$$x = \frac{n\pi}{4} \quad \text{or} \quad x = n\pi \pm \frac{\pi}{6}, \text{ where } n \in \mathbf{Z}.$$

Example 24 Solve $2 \cos^2 x + 3 \sin x = 0$

Solution The equation can be written as

$$2(1 - \sin^2 x) + 3 \sin x = 0$$

or
$$2 \sin^2 x - 3 \sin x - 2 = 0$$

or
$$(2 \sin x + 1)(\sin x - 2) = 0$$

Hence
$$\sin x = -\frac{1}{2} \quad \text{or} \quad \sin x = 2$$

But $\sin x = 2$ is not possible (Why?)

Therefore
$$\sin x = -\frac{1}{2} = \sin \frac{7\pi}{6}.$$

Hence, the solution is given by

$$x = n\pi + (-1)^n \frac{7\pi}{6}, \text{ where } n \in \mathbf{Z}.$$

EXERCISE 3.4

Find the principal and general solutions of the following equations:

- | | |
|-------------------------|----------------------------------|
| 1. $\tan x = \sqrt{3}$ | 2. $\sec x = 2$ |
| 3. $\cot x = -\sqrt{3}$ | 4. $\operatorname{cosec} x = -2$ |

Find the general solution for each of the following equations:

- | | |
|-------------------------------------|-------------------------------------|
| 5. $\cos 4x = \cos 2x$ | 6. $\cos 3x + \cos x - \cos 2x = 0$ |
| 7. $\sin 2x + \cos x = 0$ | 8. $\sec^2 2x = 1 - \tan 2x$ |
| 9. $\sin x + \sin 3x + \sin 5x = 0$ | |

Miscellaneous Examples

Example 25 If $\sin x = \frac{3}{5}$, $\cos y = -\frac{12}{13}$, where x and y both lie in second quadrant, find the value of $\sin(x + y)$.

Solution We know that

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \quad \dots (1)$$

Now $\cos^2 x = 1 - \sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Therefore $\cos x = \pm \frac{4}{5}$.

Since x lies in second quadrant, $\cos x$ is negative.

Hence $\cos x = -\frac{4}{5}$

Now $\sin^2 y = 1 - \cos^2 y = 1 - \frac{144}{169} = \frac{25}{169}$

i.e. $\sin y = \pm \frac{5}{13}$.

Since y lies in second quadrant, hence $\sin y$ is positive. Therefore, $\sin y = \frac{5}{13}$. Substituting the values of $\sin x$, $\sin y$, $\cos x$ and $\cos y$ in (1), we get