NCERT EXEMPLAR SELECTED PROBLEMS : PROBLEM 9 ON ITF

47. The result $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ is true when value of xy is ________.

Sol. We have $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Let $\tan^{-1} x = \alpha$ and $\tan^{-1} y = \beta$, where $\alpha, \beta \in (-\pi/2, \pi/2)$

$$\text{Now } \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$= \frac{x-y}{1+xy}$$

$$\Rightarrow \quad \tan^{-1}(\tan(\alpha - \beta)) = \tan^{-1}\frac{x-y}{1+xy} \qquad \text{(i)}$$

Now $\quad \tan^{-1}(\tan(\alpha - \beta)) = \alpha - \beta$ only if $\alpha - \beta \in (-\pi/2, \pi/2)$

Let $\quad \alpha, \beta < 0$

$\therefore \quad \alpha, \beta \in (-\pi/2, 0)$

$\therefore \quad \alpha \in (-\pi/2, 0)$ and '$-\beta$' $\in (0, \pi/2)$

$\therefore \quad \alpha - \beta \in (-\pi/2, \pi/2)$

$\Rightarrow \quad \tan^{-1}(\tan(\alpha - \beta)) = \alpha - \beta$

$$\Rightarrow \quad \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right) \qquad \text{(From (i))}$$

Similarly for $\alpha, \beta > 0$, we get

Similarly for $\alpha, \beta > 0$, we get

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

Let $\quad \alpha > 0$ and $\beta < 0$

$\therefore \quad \alpha$, '$-\beta$' $\in (0, \pi/2)$

$\therefore \quad \alpha - \beta \in (0, \pi)$

But we must have $\alpha - \beta \in (0, \pi/2)$

$\Rightarrow \quad \alpha - \beta < \pi/2$

$\Rightarrow \quad \alpha < \pi/2 + \beta$

$\Rightarrow \quad \tan\alpha < \tan(\pi/2 + \beta)$

$\Rightarrow \quad \tan\alpha < -\cot\beta$

$\Rightarrow \quad \tan\alpha < -\dfrac{1}{\tan\beta}$

$\Rightarrow \quad \tan\alpha \tan\beta > -1 \qquad$ (as $\tan\beta < 0$)

$\Rightarrow \quad xy > -1$

Similarly we get condition $xy > -1$ when $\alpha < 0$ and $\beta > 0$