NCERT EXEMPLAR SELECTED PROBLEMS : PROBLEM 3 ON ITF

If $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression with common difference d, then evaluate the following expression.

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2\,a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3a_4}\right)+\cdots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]$$

Since $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression with common difference d

$$d=a_2-a_1=a_3-a_2=a_4-a_3=\ldots\ldots=a_n-a_{n-1}$$

$$\therefore \quad \tan^{-1}\frac{d}{1+a_1a_2}=\tan^{-1}\frac{a_2-a_1}{1+a_1a_2}=\tan^{-1}a_2-\tan^{-1}a_1$$

Similarly $\tan^{-1}\dfrac{d}{1+a_2a_3}=\tan^{-1}\dfrac{a_3-a_2}{1+a_2a_3}=\tan^{-1}a_3-\tan^{-1}a_2$

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$$\tan^{-1}\frac{d}{1+a_{n-1}a_n}=\tan^{-1}\frac{a_n-a_{n-1}}{1+a_{n-1}a_n}=\tan^{-1}a_n-\tan^{-1}a_{n-1}$$

$$\therefore \quad \tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3a_4}\right)+\cdots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]$$

$$=\tan[(\tan^{-1}a_2-\tan^{-1}a_1)+(\tan^{-1}a_3-\tan^{-1}a_2)+\ldots+(\tan^{-1}a_n-\tan^{-1}a_{n-1})]$$

$$=\tan[\tan^{-1}a_n-\tan^{-1}a_1]$$

$$=\tan\left[\tan^{-1}\frac{a_n-a_1}{1+a_na_1}\right] \quad \left[\because \tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$=\frac{a_n-a_1}{1+a_na_1} \quad [\because \tan(\tan^{-1}x)=x]$$