For any positive integer n, let $S_n : (0, \infty) \rightarrow \mathbb{R}$ be defined by

$$S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1 + k(k+1)x^2}{x}\right),$$

where for any $x \in \mathbb{R}$, $\cot^{-1}(x) \in (0, \pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) **TRUE** ?

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$

(B) $\lim_{n\to\infty} \cot(S_n(x)) = x$, for all $x > 0$

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$

(D) $\tan(S_n(x)) \leq \frac{1}{2}$, for all $n \geq 1$ and $x > 0$

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Sol:→

$$S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right)$$

$$= \sum_{k=1}^{n} \tan^{-1}\left(\frac{x}{1+k(k+1)x^2}\right) \qquad \left(\because \cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right)\right)$$

$$= \sum_{k=1}^{n} \tan^{-1}\left(\frac{(k+1)x - (k)x}{1+k(k+1)x^2}\right) \qquad \left(\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}(x) - \tan^{-1}(y)\right)$$

$$= \sum_{k=1}^{n} \left[\tan^{-1}((k+1)x) - \tan^{-1}(kx)\right]$$

$$= \cancel{\tan^{-1}(2x)} - \tan^{-1}(x)$$
$$+ \cancel{\tan^{-1}(3x)} - \cancel{\tan^{-1}(2x)}$$
$$+ \cancel{\tan^{-1}(4x)} - \cancel{\tan^{-1}(3x)}$$
$$\vdots$$
$$\tan^{-1}((n+1)x) - \cancel{\tan^{-1}(nx)}$$

$$\boxed{S_n(x) = \tan^{-1}(n+1)x - \tan^{-1}(x)}$$

(A) $S_{10}(x) = \tan^{-1}(11x) - \tan^{-1}(x) = \tan^{-1}\left(\frac{11x - x}{1+11x^2}\right) = \tan^{-1}\left(\frac{10x}{1+11x^2}\right)$

$\therefore S_{10}(x) = \tan^{-1}\left(\frac{10x}{1+11x^2}\right) = \frac{\pi}{2} - \cot^{-1}\left(\frac{10x}{1+11x^2}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$

Hence, (A) is correct.

(B) $\lim_{n\to\infty} \cot(S_n(x)) = \lim_{n\to\infty} \cot\left(\tan^{-1}(n+1)x - \tan^{-1}(x)\right)$

$= \lim_{n\to\infty} \cot\left(\tan^{-1}\left(\frac{(n+1)x - x}{1+(n+1)x^2}\right)\right)$

$= \lim_{n\to\infty} \cot\left(\tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)\right)$

$= \lim_{n\to\infty} \cot\left(\tan^{-1}\left(\frac{x}{\frac{1}{n}+\left(1+\frac{1}{n}\right)x^2}\right)\right)$

$= \cot\left(\tan^{-1}\left(\frac{x}{x^2}\right)\right)$ as $n\to\infty$, $\frac{1}{n}\to 0$

$= \cot\left(\cot^{-1}\left(\frac{x^2}{x}\right)\right) = x$

Hence, (B) is correct.

(C) $S_3(x) = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}(4x) - \tan^{-1}(x) = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{3x}{1+4x^2}\right) = \tan^{-1}(1)$

$\Rightarrow 3x = 1+4x^2$

$\Rightarrow \boxed{4x^2 - 3x + 1 = 0} \longrightarrow D<0$
No real root

(C) is incorrect

(D) $\tan(S_n(x)) = \tan\left(\tan^{-1}(n+1)x - \tan^{-1}(x)\right) = \tan\left(\tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)\right)$

$\therefore \tan(S_n(x)) = \frac{nx}{1+(n+1)x^2}$ $\quad (\because \tan(\tan^{-1}(y)) = y \ \forall y \in R)$

$\therefore \quad \tan(S_n(x)) = \dfrac{1}{\frac{1}{nx} + \frac{(n+1)x^2}{nx}} = \dfrac{1}{\frac{1}{nx} + x + \frac{x}{n}}$

as $n \to \infty$, $\dfrac{1}{\frac{1}{nx} + x + \frac{x}{n}} = \dfrac{1}{x}$; Say $x = 1$

Value $= 1$

$\therefore \quad \tan(S_n(x))$ is not $\leq \frac{1}{2} \; \forall \, n \geq 1$ and $x > 0$

(D) is incorrect.

Hence correct ans is (AB)