Q.3 For non-negative integers n, let

$$f(n)=\frac{\sum_{k=0}^{n}\sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n}\sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming $\cos^{-1}x$ takes values in $[0,\pi]$, which of the following options is/are correct?

(A) $f(4)=\frac{\sqrt{3}}{2}$

(B) $\lim_{n\to\infty} f(n)=\frac{1}{2}$

(C) If $\alpha=\tan(\cos^{-1}f(6))$, then $\alpha^2+2\alpha-1=0$

(D) $\sin(7\cos^{-1}f(5))=0$

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Solⁿ:→

$$f(n)=\frac{\sum_{k=0}^{n}2\sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n}2\sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

(Multiply N^r & D^r by 2)

Using $2\sin A\sin B = \cos(A-B)-\cos(A+B)$

$$\therefore\ 2\sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)=\cos\left(\frac{\pi}{n+2}\right)-\cos\left(\left(\frac{2k+3}{n+2}\right)\pi\right)$$

Also $$2\sin^2\left(\frac{k+1}{n+2}\pi\right)=1-\cos\left(\left(\frac{2k+2}{n+2}\right)\pi\right)$$

$$\therefore\ f(n)=\frac{\sum_{k=0}^{n}\cos\left(\frac{\pi}{n+2}\right)-\cos\left(\left(\frac{2k+3}{n+2}\right)\pi\right)}{\sum_{k=0}^{n}1-\cos\left(\left(\frac{2k+2}{n+2}\right)\pi\right)}$$

$$\Rightarrow f(n)=\frac{(n+1)\cos\left(\frac{\pi}{n+2}\right)-\sum_{k=0}^{n}\cos\left(\left(\frac{2k+3}{n+2}\right)\pi\right)\ \text{—(I)}}{(n+1)-\sum_{k=0}^{n}\cos\left(\left(\frac{2k+2}{n+2}\right)\pi\right)\ \text{—(II)}}$$

(I) $\sum_{k=0}^{n} \cos\left(\frac{2k+3}{n+2}\right)\pi = \cos\left(\frac{3\pi}{n+2}\right) + \cos\left(\frac{5\pi}{n+2}\right) + \cdots + \cos\left(\left(\frac{2n+3}{n+2}\right)\pi\right)$

$$\left[\cos(\theta) + \cos(\theta+\beta) + \cos(\theta+2\beta) + \cdots + \cos(\theta+(n-1)\beta) = \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cos\left(\theta + \frac{(n-1)\beta}{2}\right)\right]$$

$$\therefore \sum_{k=0}^{n} \cos\left(\frac{2k+3}{n+2}\right)\pi = \frac{\sin\left(\frac{(n+1)\pi}{(n+2)}\right)}{\sin\left(\frac{\pi}{n+2}\right)} \cos\left(\frac{3\pi}{n+2} + \frac{n\pi}{n+2}\right)$$

$$\boxed{\sum_{k=0}^{n} \cos\left(\frac{2k+3}{n+2}\right)\pi = \frac{\sin (n+1)\frac{\pi}{n+2}}{\sin\left(\frac{\pi}{n+2}\right)} \cos\left(\left(\frac{n+3}{n+2}\right)\pi\right)} \quad \text{—— (I.)}$$

(II.) $\sum_{k=0}^{n} \cos\left(\frac{2k+2}{n+2}\right)\pi = \cos\left(\frac{2\pi}{n+2}\right) + \cos\left(\frac{4\pi}{n+2}\right) + \cdots + \cos\left(\left(\frac{2n+2}{n+2}\right)\pi\right)$

$$\therefore \sum_{k=0}^{n} \cos\left(\frac{2k+2}{n+2}\right)\pi = \frac{\sin\left(\frac{(n+1)\pi}{(n+2)}\right)}{\sin\left(\frac{\pi}{n+2}\right)} \cos\left(\frac{2\pi}{n+2} + \frac{n\pi}{n+2}\right)$$

$$\therefore \boxed{\sum_{k=0}^{n} \cos\left(\frac{2k+2}{n+2}\right)\pi = \frac{\sin\left(\frac{(n+1)\pi}{n+2}\right)}{\sin\left(\frac{\pi}{n+2}\right)} \cos(\pi)} \quad \text{—— (II.)}$$

So
$$f(n) = \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) - \dfrac{\sin\frac{(n+1)\pi}{(n+2)}}{\sin\left(\frac{\pi}{n+2}\right)} \cos\left(\frac{n+3}{n+2}\right)\pi}{(n+1) + \dfrac{\sin\left(\frac{(n+1)\pi}{(n+2)}\right)}{\sin\left(\frac{\pi}{n+2}\right)}}$$

$\sin(\pi - \theta) = \sin(\theta)$; $\cos(\pi+\theta) = -\cos(\theta)$

Using above properties,

$$f(n) = \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) + \cos\left(\frac{\pi}{n+2}\right)}{(n+2)}$$

$$\boxed{f(n) = \cos\left(\frac{\pi}{n+2}\right)}$$

As an advise, the problem looks complicated and difficult to solve. But the tip here should be to solve individual parts of the problem separately, like I have tried to solve. These kind of problems do come in JEE Advanced, especially the kind of problems where we have to solve kind of summation.

a) $f(4) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ (A) is correct

b) $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \cos\left(\frac{\pi}{n+2}\right) = \cos(0) = 1$ (B) is incorrect

c) $\alpha = \tan\left(\cos^{-1}(f(6))\right)$; $f(6) = \cos\left(\frac{\pi}{8}\right)$

$\cos^{-1}(f(6)) = \cos^{-1}\left(\cos\left(\pi/8\right)\right) = \pi/8$

$\alpha = \tan\left(\pi/8\right) = \sqrt{2}-1$

$\alpha^2 = 3-2\sqrt{2}$

$\alpha^2+2\alpha-1 = 3-\cancel{2\sqrt{2}}+\cancel{2\sqrt{2}}-2-1 = 0$ (C) is correct

d) $\sin\left(7\cos^{-1}(f(5))\right) = \sin\left(7.\cos^{-1}\left(\cos\frac{\pi}{7}\right)\right) = \sin\left(7.\pi/7\right) = 0$

(D) is correct

These kind of questions can be scoring, if you keep your calm in the exam. As you can see, once solved the options are all one liners.