**12.** If $y = \cos^{-1} x$, Find $\frac{d^2y}{dx^2}$ in terms of $y$ alone.

**13.** If $y = 3 \cos (\log x) + 4 \sin (\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

**14.** If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

**15.** If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

**16.** If $e^y (x + 1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

**17.** If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x (x^2 + 1) y_1 = 2$

## 5.8 Mean Value Theorem

In this section, we will state two fundamental results in Calculus without proof. We shall also learn the geometric interpretation of these theorems.

**Theorem 6** (Rolle's Theorem) Let $f : [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$, such that $f(a) = f(b)$, where $a$ and $b$ are some real numbers. Then there exists some $c$ in $(a, b)$ such that $f'(c) = 0$.

In Fig 5.12 and 5.13, graphs of a few typical differentiable functions satisfying the hypothesis of Rolle's theorem are given.

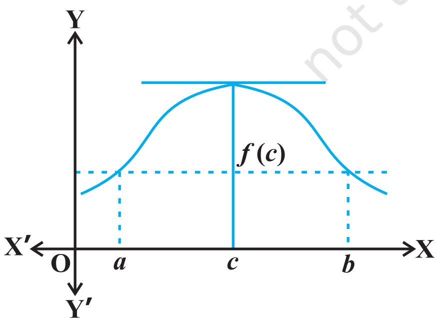


**Fig 5.12**

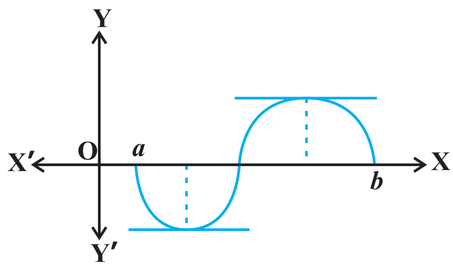


**Fig 5.13**

Observe what happens to the slope of the tangent to the curve at various points between $a$ and $b$. In each of the graphs, the slope becomes zero at least at one point. That is precisely the claim of the Rolle's theorem as the slope of the tangent at any point on the graph of $y = f(x)$ is nothing but the derivative of $f(x)$ at that point.

**Theorem 7** (Mean Value Theorem) Let $f : [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$ and differentiable on $(a, b)$. Then there exists some $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Observe that the Mean Value Theorem (MVT) is an extension of Rolle's theorem. Let us now understand a geometric interpretation of the MVT. The graph of a function $y = f(x)$ is given in the Fig 5.14. We have already interpreted $f'(c)$ as the slope of the tangent to the curve $y = f(x)$ at $(c, f(c))$. From the Fig 5.14 it is clear that $\frac{f(b) - f(a)}{b - a}$ is the slope of the secant drawn between $(a, f(a))$ and $(b, f(b))$. The MVT states that there is a point $c$ in $(a, b)$ such that the slope of the tangent at $(c, f(c))$ is same as the slope of the secant between $(a, f(a))$ and $(b, f(b))$. In other words, there is a point $c$ in $(a, b)$ such that the tangent at $(c, f(c))$ is parallel to the secant between $(a, f(a))$ and $(b, f(b))$.

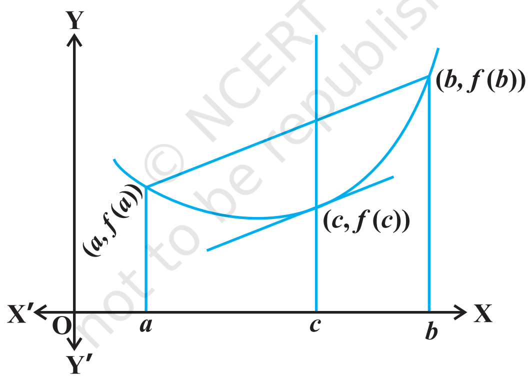


**Fig 5.14**

**Example 42** Verify Rolle's theorem for the function $y = x^2 + 2$, $a = -2$ and $b = 2$.

**Solution** The function $y = x^2 + 2$ is continuous in $[-2, 2]$ and differentiable in $(-2, 2)$. Also $f(-2) = f(2) = 6$ and hence the value of $f(x)$ at $-2$ and $2$ coincide. Rolle's theorem states that there is a point $c \in (-2, 2)$, where $f'(c) = 0$. Since $f'(x) = 2x$, we get $c = 0$. Thus at $c = 0$, we have $f'(c) = 0$ and $c = 0 \in (-2, 2)$.

**Example 43** Verify Mean Value Theorem for the function $f(x) = x^2$ in the interval $[2, 4]$.

**Solution** The function $f(x) = x^2$ is continuous in $[2, 4]$ and differentiable in $(2, 4)$ as its derivative $f'(x) = 2x$ is defined in $(2, 4)$.

Now, $f(2) = 4$ and $f(4) = 16$. Hence

$$\frac{f(b)-f(a)}{b-a}=\frac{16-4}{4-2}=6$$

MVT states that there is a point $c \in (2, 4)$ such that $f'(c) = 6$. But $f'(x) = 2x$ which implies $c = 3$. Thus at $c = 3 \in (2, 4)$, we have $f'(c) = 6$.

## EXERCISE 5.8

1. Verify Rolle's theorem for the function $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$.
2. Examine if Rolle's theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's theorem from these example?
   (i) $f(x) = [x]$ for $x \in [5, 9]$ (ii) $f(x) = [x]$ for $x \in [-2, 2]$
   (iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$
3. If $f : [-5, 5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.
4. Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.
5. Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.
6. Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.



## *Miscellaneous Examples*

**Example 44** Differentiate w.r.t. $x$, the following function:

(i) $\sqrt{3x+2}+\frac{1}{\sqrt{2x^2+4}}$ (ii) $e^{\sec^2 x}+3\cos^{-1}x$ (iii) $\log_7(\log x)$

**Solution**

(i) Let $y = \sqrt{3x+2}+\frac{1}{\sqrt{2x^2+4}} = (3x+2)^{\frac{1}{2}}+(2x^2+4)^{-\frac{1}{2}}$

Note that this function is defined at all real numbers $x > -\frac{2}{3}$. Therefore

$$\frac{dy}{dx}=\frac{1}{2}(3x+2)^{\frac{1}{2}-1}\cdot\frac{d}{dx}(3x+2)+\left(-\frac{1}{2}\right)(2x^2+4)^{-\frac{1}{2}-1}\cdot\frac{d}{dx}(2x^2+4)$$