

Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} = -\tan \theta = -\sqrt[3]{\frac{y}{x}}$$

EXERCISE 5.6

If x and y are connected parametrically by the equations given in Exercises 1 to 10, without eliminating the parameter, Find $\frac{dy}{dx}$.

1. $x = 2at^2, y = at^4$

2. $x = a \cos \theta, y = b \cos \theta$

3. $x = \sin t, y = \cos 2t$

4. $x = 4t, y = \frac{4}{t}$

5. $x = \cos \theta - \cos 2\theta, y = \sin \theta - \sin 2\theta$

6. $x = a(\theta - \sin \theta), y = a(1 + \cos \theta)$

7. $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

8. $x = a \left(\cos t + \log \tan \frac{t}{2} \right), y = a \sin t$

9. $x = a \sec \theta, y = b \tan \theta$

10. $x = a(\cos \theta + \theta \sin \theta), y = a(\sin \theta - \theta \cos \theta)$

11. If $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

5.7 Second Order Derivative

Let $y = f(x)$. Then

$$\frac{dy}{dx} = f'(x) \quad \dots (1)$$

If $f'(x)$ is differentiable, we may differentiate (1) again w.r.t. x . Then, the left hand side becomes $\frac{d}{dx} \left(\frac{dy}{dx} \right)$ which is called the *second order derivative* of y w.r.t. x and is denoted by $\frac{d^2 y}{dx^2}$. The second order derivative of $f(x)$ is denoted by $f''(x)$. It is also

denoted by $D^2 y$ or y'' or y_2 if $y = f(x)$. We remark that higher order derivatives may be defined similarly.

Example 38 Find $\frac{d^2 y}{dx^2}$, if $y = x^3 + \tan x$.

Solution Given that $y = x^3 + \tan x$. Then

$$\frac{dy}{dx} = 3x^2 + \sec^2 x$$

Therefore

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{d}{dx} (3x^2 + \sec^2 x) \\ &= 6x + 2 \sec x \cdot \sec x \tan x = 6x + 2 \sec^2 x \tan x \end{aligned}$$

Example 39 If $y = A \sin x + B \cos x$, then prove that $\frac{d^2 y}{dx^2} + y = 0$.

Solution We have

$$\frac{dy}{dx} = A \cos x - B \sin x$$

and

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{d}{dx} (A \cos x - B \sin x) \\ &= -A \sin x - B \cos x = -y \end{aligned}$$

Hence

$$\frac{d^2 y}{dx^2} + y = 0$$

Example 40 If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0$.

Solution Given that $y = 3e^{2x} + 2e^{3x}$. Then

$$\frac{dy}{dx} = 6e^{2x} + 6e^{3x} = 6(e^{2x} + e^{3x})$$

Therefore

$$\frac{d^2 y}{dx^2} = 12e^{2x} + 18e^{3x} = 6(2e^{2x} + 3e^{3x})$$

Hence

$$\begin{aligned} \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y &= 6(2e^{2x} + 3e^{3x}) \\ &\quad - 30(e^{2x} + e^{3x}) + 6(3e^{2x} + 2e^{3x}) = 0 \end{aligned}$$

Example 41 If $y = \sin^{-1} x$, show that $(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = 0$.

Solution We have $y = \sin^{-1} x$. Then

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

or
$$\sqrt{1-x^2} \frac{dy}{dx} = 1$$

So
$$\frac{d}{dx} \left(\sqrt{1-x^2} \cdot \frac{dy}{dx} \right) = 0$$

or
$$\sqrt{1-x^2} \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot \frac{d}{dx} \left(\sqrt{1-x^2} \right) = 0$$

or
$$\sqrt{1-x^2} \cdot \frac{d^2 y}{dx^2} - \frac{dy}{dx} \cdot \frac{2x}{2\sqrt{1-x^2}} = 0$$

Hence
$$(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = 0$$

Alternatively, Given that $y = \sin^{-1} x$, we have

$$y_1 = \frac{1}{\sqrt{1-x^2}}, \text{ i.e., } (1-x^2) y_1^2 = 1$$

So
$$(1-x^2) \cdot 2y_1 y_2 + y_1^2 (0-2x) = 0$$

Hence
$$(1-x^2) y_2 - x y_1 = 0$$

EXERCISE 5.7

Find the second order derivatives of the functions given in Exercises 1 to 10.

1. $x^2 + 3x + 2$

2. x^{20}

3. $x \cdot \cos x$

4. $\log x$

5. $x^3 \log x$

6. $e^x \sin 5x$

7. $e^{6x} \cos 3x$

8. $\tan^{-1} x$

9. $\log(\log x)$

10. $\sin(\log x)$

11. If $y = 5 \cos x - 3 \sin x$, prove that $\frac{d^2 y}{dx^2} + y = 0$