Handwritten Notes

On

Definite Integrals

Definite Integral

* Definite Integral as the limit of a sum:

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty} \frac{1}{n}\left[f(a)+f(a+h)+\dots+f(a+(n-1)h)\right]$$

where $h = \frac{b-a}{n} \to 0$ as $n\to\infty$

* Let f be a continuous function on the closed interval $[a,b]$ and let $A(x)$ be the area function. Then $A'(x) = f(x) \ \forall\ x \in [a,b]$.

* Let f be a continuous function defined on the closed interval $[a,b]$ and F be an antiderivative of f. Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

* In $\int_a^b f(x)\,dx$, the function f needs to be well defined & continuous in $[a,b]$.

* Evaluation of definite integral:

i) by the theorem $\int_a^b f(x)\,dx = F(b) - F(a)$ by finding antiderivative F.

ii) by substitution

* Properties (Proofs*)

i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. | $\int_a^a f(x)\,dx = 0$.

iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. $(a<c<b)$

iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.

v) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

vi) $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx.$

vii) $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \quad , \text{ if } \quad f(2a-x) = f(x)$

$= 0 \quad , \text{ if } \quad f(2a-x) = -f(x).$

viii) $\int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$, if f is even

$= 0$, if f is odd.

ix) If $f(x)$ is a periodic function with period T, then $\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx.$

x) $\int_a^{a+nT} f(x)\,dx = n\int_0^{T} f(x)\,dx.$

xi) $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^{T} f(x)\,dx$

xii) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^{b} f(x)\,dx$

xiii) Leibnitz's Rule: If $\phi(x)$ & $\psi(x)$ are defined on $[a,b]$ & are differentiable at a point $x \in (a,b)$ and $f(x,t)$ is continuous, then

a) $\dfrac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)\,dt\right] = \int_{\phi(x)}^{\psi(x)} \dfrac{\partial}{\partial x} f(x,t)\,dt + \left\{\dfrac{d\psi(x)}{dx}\right\} f(x,\psi(x)) - \left\{\dfrac{d\phi(x)}{dx}\right\} f(x,\phi(x))$

b) $\dfrac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right) = \dfrac{d}{dx}\{\psi(x)\}\, f(\psi(x)) - \dfrac{d}{dx}\{\phi(x)\}\, f(\phi(x)).$

xiv) Let a function $f(x,\alpha)$ be continuous for $a \le x \le b$ & $c \le \alpha \le d$, then for any $\alpha \in [c,d]$, if $I(\alpha) = \int_a^{b} f(x,\alpha)\,dx$, then

$$\frac{dI(\alpha)}{d\alpha} = \int_a^{b} \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx.$$

* If $\int_a^{b} f(x)\,dx = 0$ then $f(x) = 0$ has at least one root lying in (a,b) provided f is continuous function in (a,b).

xv) $\int_a^b f(x)\,dx = (b-a)\int_0^1 f((b-a)x+a)\,dx$.

xvi) If $f(x)$ is defined on $[a,b]$, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$ [Equality holds where $f(x)$ is entirely of some sign on $[a,b]$].

xvii) $$\left|\int_a^b f(x)g(x)\,dx\right| \le \sqrt{\left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right)}$$ → Schwarz-Bunyakovsky inequality.

xviii) If $f(x) \geq g(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

xix) $f(x)$ continuous on $[a,b]$, $f_1(x)$ & $f_2(x)$ such that $f_1(x) \le f(x) \le f_2(x)$ $\forall\ x \in [a,b]$ then

$$\int_a^b f_1(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f_2(x)\,dx.$$

xx) If m & M be global minimum & global maximum of $f(x)$ respectively in $[a,b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

*xxi) If $f(t)$ is an odd function, then $\phi(x) = \int_a^x f(t)\,dt$ is an even function.

xxii) If $f(t)$ is an even function, then $\phi(x) = \int_0^x f(t)\,dt$ is an odd function.

xxiii) If $f(x)$ is continuous on $[a,\infty]$ then $\int_a^\infty f(x)\,dx$ is called an improper integral and is defined as $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$.

If there exists a finite limit on the rhs then the improper integral is convergent, otherwise divergent.

- Geometrically, for $f(x) > 0$ the improper integral $\int_a^\infty f(x)\,dx$ gives area of the figure bounded by the curve $y=f(x)$ & the x axis & $x=a$ st. line.
- $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} + \int_a^b f(x)\,dx$.
- $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^{+\infty} f(x)\,dx$.

* Mean Value of function: $f(x)$ continuous on $[a,b]$, then there exists a point $c \in (a,b)$ s.t. $\int_a^b f(x)dx = f(c)\,(b-a)$

$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ is the mean value of $f(x)$ over $[a,b]$.

* Method to express infinite series as definite integral: i) Expressing the series in form $\Sigma \frac{1}{n} f\left(\frac{r}{n}\right)$.

ii) the sum is $\lim_{n\to\infty} \Sigma \frac{1}{n} f\left(\frac{r}{n}\right)$.

iii) replacing r/n by x & $1/n$ by (dx) & $\lim_{n\to\infty}\Sigma$ by sign '$\int$'.

iv) lower & upper limit are limiting values of r/n for the first and last term of r respectively.

- **Particular Cases:**

a) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right)$ or $\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx.$

b) $\lim_{n\to\infty} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_\alpha^\beta f(x)\,dx$ | $\alpha = \lim_{n\to\infty} \frac{r}{n} = 0$

$\beta = \lim_{n\to\infty} \frac{r}{n} = p.$

* Walli's formula: $\int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx = \int_0^{\pi/2} \sin^n x \cdot \cos^m x\,dx =$

$$\frac{(m-1)(m-3)\dots(1 \text{ or } 2)\cdot(n-1)(n-3)\dots(1 \text{ or } 2)}{(m+n)(m+n-2)\dots(1 \text{ or } 2)}\cdot\frac{\pi}{2}.$$

[when both m & $n \in$ even integer]

$$\frac{(m-1)(m-3)\dots(1 \text{ or } 2)\cdot(n-1)(n-3)(n-5)\dots(1 \text{ or } 2)}{(m+n)(m+n-2)\dots(1 \text{ or } 2)}$$

[when either of m or $n \in$ odd I].

- If n be a +ve I, then

$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx =$

$\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\dots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$ (n is even)

$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\dots\frac{4}{5}\cdot\frac{2}{3}\cdot 1$, ($n$ is odd).

Important results -

i) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

ii) $\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1).$

iii) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}.$

iv) In GP, sum of n terms, $S_n = \begin{cases} \frac{a(r^n-1)}{r-1}, & |r| > 1 \\ an, & r = 1 \\ \frac{a(1-r^n)}{1-r}, & |r| < 1 \end{cases}$

v) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots + \sin(\alpha + \overline{n-1}\,\beta)$

$$= \frac{\sin n\beta/2}{\sin \beta/2} \cdot \sin(\alpha + (n-1)\beta/2).$$

vi) $\cos\alpha + \cos(\alpha+\beta) + \dots + \cos(\alpha + \overline{n-1}\,\beta) =$

$$\frac{\sin n\beta/2}{\sin\beta/2} \cdot \cos(\alpha + (n-1)\beta/2).$$

vii) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots \infty = \frac{\pi^2}{12}.$

viii) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \infty = \frac{\pi^2}{6}.$

ix) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}.$

x) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots \infty = \frac{\pi^2}{24}.$

xi) $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

xii) $\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}, \quad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}.$

xiii) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \infty = \ln 2$