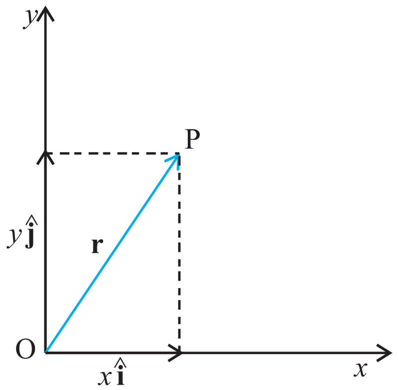
4.7.1 Position Vector and Displacement

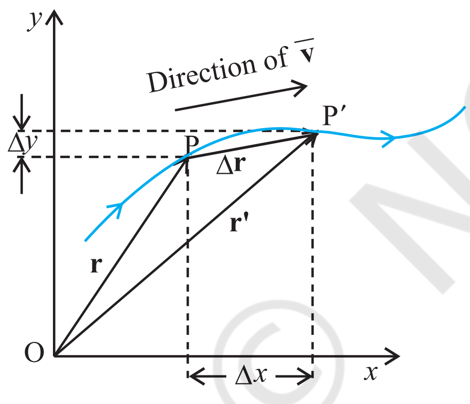
The position vector $\mathbf{r}$ of a particle P located in a plane with reference to the origin of an x-y reference frame (Fig. 4.12) is given by

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$$

where x and y are components of $\mathbf{r}$ along x-, and y- axes or simply they are the coordinates of the object.



(a)



(b)

Fig. 4.12 *(a) Position vector* $\mathbf{r}$. *(b) Displacement* $\Delta\mathbf{r}$ *and average velocity* $\bar{\mathbf{v}}$ *of a particle.*

Suppose a particle moves along the curve shown by the thick line and is at P at time t and P′ at time t' [Fig. 4.12(b)]. Then, the displacement is :

$$\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r} \tag{4.25}$$

and is directed from P to P′ .

We can write Eq. (4.25) in a component form:

$$\Delta\mathbf{r} = \left(x'\,\hat{\mathbf{i}} + y'\,\hat{\mathbf{j}}\right) - \left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}\right)$$
$$= \hat{\mathbf{i}}\Delta x + \hat{\mathbf{j}}\Delta y$$

where $\Delta x = x' - x,\ \Delta y = y' - y$ (4.26)

Velocity

The average velocity $(\bar{\mathbf{v}})$ of an object is the ratio of the displacement and the corresponding time interval :

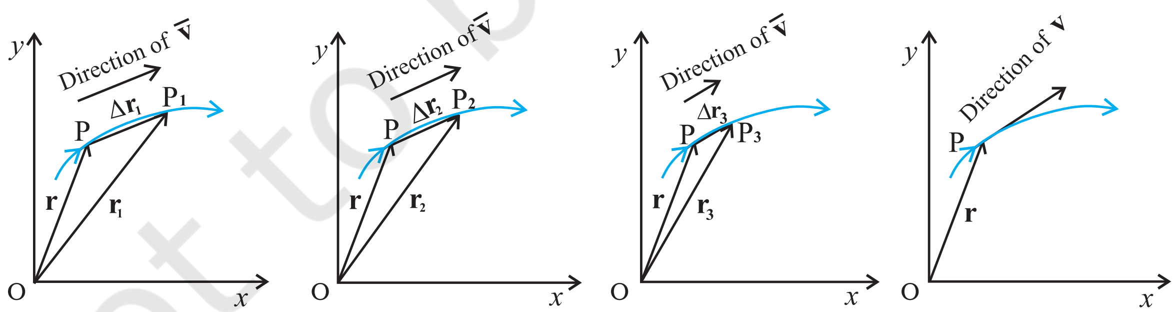
$$\bar{\mathbf{v}} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\Delta x\,\hat{\mathbf{i}} + \Delta y\,\hat{\mathbf{j}}}{\Delta t} = \hat{\mathbf{i}}\frac{\Delta x}{\Delta t} + \hat{\mathbf{j}}\frac{\Delta y}{\Delta t} \tag{4.27}$$

Or, $\bar{\mathbf{v}} = \bar{v}_x\,\hat{\mathbf{i}} + \bar{v}_y\,\hat{\mathbf{j}}$

Since $\bar{\mathbf{v}} = \dfrac{\Delta\mathbf{r}}{\Delta t}$, the direction of the average velocity is the same as that of $\Delta\mathbf{r}$ (Fig. 4.12). The **velocity** (instantaneous velocity) is given by the limiting value of the average velocity as the time interval approaches zero :

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \tag{4.28}$$

The meaning of the limiting process can be easily understood with the help of Fig 4.13(a) to (d). In these figures, the thick line represents the path of an object, which is at P at time t. P_1, P_2 and P_3 represent the positions of the object after times Δt_1, Δt_2, and Δt_3. $\Delta\mathbf{r}_1$, $\Delta\mathbf{r}_2$, and $\Delta\mathbf{r}_3$ are the displacements of the object in times Δt_1, Δt_2, and



(a) (b) (c) (d)

Fig. 4.13 *As the time interval* Δt *approaches zero, the average velocity approaches the velocity* $\mathbf{v}$. *The direction of* $\bar{\mathbf{v}}$ *is parallel to the line tangent to the path.*

Δt_3, respectively. The direction of the average velocity $\overline{\mathbf{v}}$ is shown in figures (a), (b) and (c) for three decreasing values of Δt, i.e. Δt_1,Δt_2, and Δt_3, ($\Delta t_1 > \Delta t_2 > \Delta t_3$). As $\Delta t \to 0$, $\Delta \mathbf{r} \to 0$ and is along the tangent to the path [Fig. 4.13(d)]. Therefore, **the direction of velocity at any point on the path of an object is tangential to the path at that point and is in the direction of motion**.

We can express **v** in a component form :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$= \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta y}{\Delta t}\hat{\mathbf{j}}\right) \qquad (4.29)$$

$$= \hat{\mathbf{i}} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \hat{\mathbf{j}} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

Or, $$\mathbf{v} = \hat{\mathbf{i}}\frac{dx}{dt} + \hat{\mathbf{j}}\frac{dy}{dt} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}.$$

where $$v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt} \qquad (4.30a)$$

So, if the expressions for the coordinates x and y are known as functions of time, we can use these equations to find v_x and v_y.

The magnitude of **v** is then

$$v = \sqrt{v_x^2 + v_y^2} \qquad (4.30b)$$

and the direction of **v** is given by the angle θ :

$$\tan\theta = \frac{v_y}{v_x}, \quad \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \qquad (4.30c)$$

v_x, v_y and angle θ are shown in Fig. 4.14 for a velocity vector **v** at point **p**.

Acceleration

The **average acceleration** **a** of an object for a time interval Δt moving in x-y plane is the change in velocity divided by the time interval :

$$\overline{\mathbf{a}} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\Delta\left(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}\right)}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta v_y}{\Delta t}\hat{\mathbf{j}} \qquad (4.31a)$$

Or, $$\overline{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}. \qquad (4.31b)$$

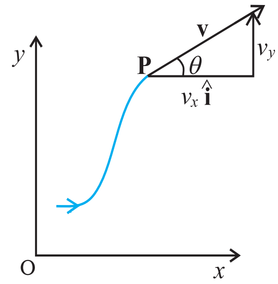


Fig. 4.14 *The components v_x and v_y of velocity **v** and the angle θ it makes with x-axis. Note that $v_x = v\cos\theta$, $v_y = v\sin\theta$.*

The **acceleration** (instantaneous acceleration) is the limiting value of the average acceleration as the time interval approaches zero :

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t} \qquad (4.32a)$$

Since $\Delta\mathbf{v} = \Delta v_x\hat{\mathbf{i}} + \Delta v_y\hat{\mathbf{j}}$,we have

$$\mathbf{a} = \hat{\mathbf{i}} \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} + \hat{\mathbf{j}} \lim_{\Delta t \to 0} \frac{\Delta v_y}{\Delta t}$$

Or, $$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} \qquad (4.32b)$$

where, $$a_x = \frac{dv_x}{dt}, \; a_y = \frac{dv_y}{dt} \qquad (4.32c)^*$$

As in the case of velocity, we can understand graphically the limiting process used in defining acceleration on a graph showing the path of the object's motion. This is shown in Figs. 4.15(a) to (d). P represents the position of the object at time t and P_1, P_2, P_3 positions after time Δt_1, Δt_2, Δt_3, respectively ($\Delta t_1 > \Delta t_2 > \Delta t_3$). The velocity vectors at points P, P_1, P_2, P_3 are also shown in Figs. 4.15 (a), (b) and (c). In each case of Δt, $\Delta\mathbf{v}$ is obtained using the triangle law of vector addition. By definition, the direction of average acceleration is the same as that of $\Delta\mathbf{v}$. We see that as Δt decreases, the direction of $\Delta\mathbf{v}$ changes and consequently, the direction of the acceleration changes. Finally, in the limit $\Delta t \to 0$ [Fig. 4.15(d)], the average acceleration becomes the instantaneous acceleration and has the direction as shown.

* *In terms of x and y, a_x and a_y can be expressed as*

$$a_x = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}, \; a_y = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

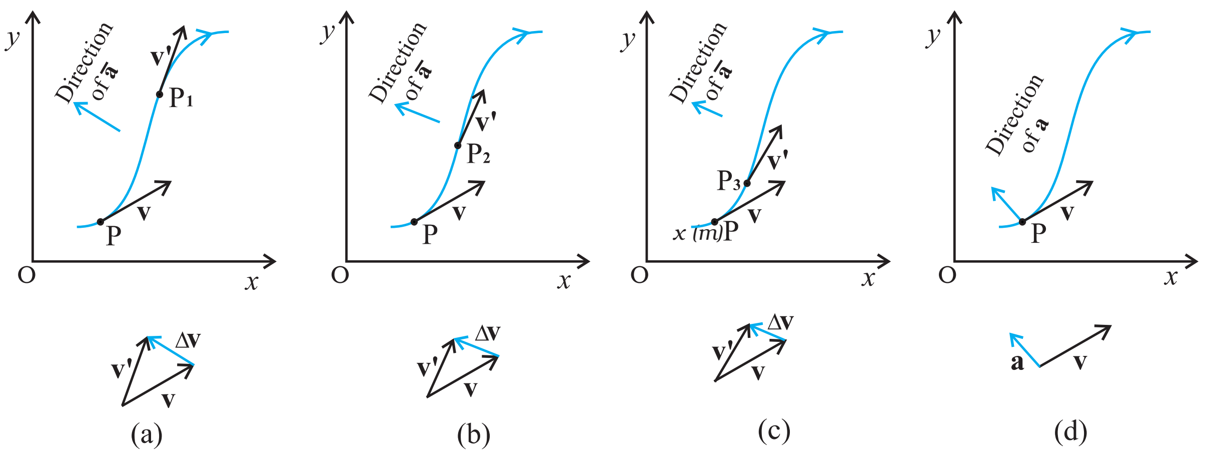


Fig. 4.15 *The average acceleration for three time intervals* (a) Δt_1, (b) Δt_2, *and* (c) Δt_3, $(\Delta t_1 > \Delta t_2 > \Delta t_3)$. (d) *In the limit* $\Delta t \rightarrow 0$, *the average acceleration becomes the acceleration.*

Note that in one dimension, the velocity and the acceleration of an object are always along the same straight line (either in the same direction or in the opposite direction). However, for motion in two or three dimensions, velocity and acceleration vectors may have any angle between 0° and 180° between them.

▶ ***Example 4.4*** The position of a particle is given by

$$\mathbf{r} = 3.0t\,\hat{\mathbf{i}} + 2.0t^2\,\hat{\mathbf{j}} + 5.0\,\hat{\mathbf{k}}$$

where t is in seconds and the coefficients have the proper units for $\mathbf{r}$ to be in metres. (a) Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$ of the particle. (b) Find the magnitude and direction of $\mathbf{v}(t)$ at $t = 1.0$ s.

Answer

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(3.0\,t\,\hat{\mathbf{i}} + 2.0t^2\,\hat{\mathbf{j}} + 5.0\,\hat{\mathbf{k}}\right)$$

$$= 3.0\hat{\mathbf{i}} + 4.0t\hat{\mathbf{j}}$$

$$\mathbf{a}\,(t) = \frac{d\mathbf{v}}{dt} = +4.0\hat{\mathbf{j}}$$

$a = 4.0$ m s^{-2} along y- direction

At $t = 1.0$ s, $\mathbf{v} = 3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}}$

It's magnitude is $v = \sqrt{3^2 + 4^2} = 5.0$ m s^{-1}

and direction is

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) \cong 53^\circ \text{ with } x\text{-axis.}$$

◀

4.8 MOTION IN A PLANE WITH CONSTANT ACCELERATION

Suppose that an object is moving in x-y plane and its acceleration $\mathbf{a}$ is constant. Over an interval of time, the average acceleration will equal this constant value. Now, let the velocity of the object be $\mathbf{v}_0$ at time $t = 0$ and $\mathbf{v}$ at time t.

Then, by definition

$$\mathbf{a} = \frac{\mathbf{v} - \mathbf{v}_0}{t - 0} = \frac{\mathbf{v} - \mathbf{v}_0}{t}$$

Or, $\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t$ (4.33a)

In terms of components :

$$v_x = v_{ox} + a_x t$$

$$v_y = v_{oy} + a_y t \qquad (4.33b)$$

Let us now find how the position $\mathbf{r}$ changes with time. We follow the method used in the one-dimensional case. Let $\mathbf{r}_o$ and $\mathbf{r}$ be the position vectors of the particle at time 0 and t and let the velocities at these instants be $\mathbf{v}_o$ and $\mathbf{v}$. Then, over this time interval t, the average velocity is $(\mathbf{v}_o + \mathbf{v})/2$. The displacement is the average velocity multiplied by the time interval :

$$\mathbf{r} - \mathbf{r}_0 = \left(\frac{\mathbf{v} + \mathbf{v}_0}{2}\right)t = \left(\frac{(\mathbf{v}_0 + \mathbf{a}t) + \mathbf{v}_0}{2}\right)t$$

$$= \mathbf{v_0}t + \frac{1}{2}\mathbf{a}t^2$$

Or, $$\mathbf{r} = \mathbf{r_0} + \mathbf{v_0}t + \frac{1}{2}\mathbf{a}t^2 \quad (4.34a)$$

It can be easily verified that the derivative of Eq. (4.34a), i.e. $\frac{d\mathbf{r}}{dt}$ gives Eq.(4.33a) and it also satisfies the condition that at t=0, $\mathbf{r} = \mathbf{r_o}$. Equation (4.34a) can be written in component form as

$$x = x_0 + v_{ox}t + \frac{1}{2}a_x t^2$$

$$y = y_0 + v_{oy}t + \frac{1}{2}a_y t^2 \quad (4.34b)$$

One immediate interpretation of Eq.(4.34b) is that the motions in x- and y-directions can be treated independently of each other. That is, **motion in a plane (two-dimensions) can be treated as two separate simultaneous one-dimensional motions with constant acceleration along two perpendicular directions**. This is an important result and is useful in analysing motion of objects in two dimensions. A similar result holds for three dimensions. The choice of perpendicular directions is convenient in many physical situations, as we shall see in section 4.10 for projectile motion.

Example 4.5 A particle starts from origin at $t = 0$ with a velocity $5.0\,\hat{\mathbf{i}}$ m/s and moves in x-y plane under action of a force which produces a constant acceleration of $(3.0\hat{\mathbf{i}}+2.0\hat{\mathbf{j}})$ m/s². (a) What is the y-coordinate of the particle at the instant its x-coordinate is 84 m ? (b) What is the speed of the particle at this time ?

Answer From Eq. (4.34a) for $\mathbf{r_0} = 0$, the position of the particle is given by

$$\mathbf{r}(t) = \mathbf{v_0}t + \frac{1}{2}\mathbf{a}t^2$$

$$= 5.0\hat{\mathbf{i}}\,t + (1/2)\left(3.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}}\right)t^2$$

$$= \left(5.0t + 1.5t^2\right)\hat{\mathbf{i}} + 1.0t^2\hat{\mathbf{j}}$$

Therefore, $$x(t) = 5.0t + 1.5t^2$$

$$y(t) = +1.0t^2$$

Given $x(t) = 84$ m, $t = ?$

$$5.0\,t + 1.5\,t^2 = 84 \Rightarrow t = 6 \text{ s}$$

At $t = 6$ s, $y = 1.0\,(6)^2 = 36.0$ m

Now, the velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (5.0 + 3.0t)\hat{\mathbf{i}} + 2.0t\,\hat{\mathbf{j}}$

At $t = 6$ s, $\mathbf{v} = 23.0\hat{\mathbf{i}} + 12.0\hat{\mathbf{j}}$

speed $= |\mathbf{v}| = \sqrt{23^2 + 12^2} \cong 26 \text{ m s}^{-1}$.

4.9 RELATIVE VELOCITY IN TWO DIMENSIONS

The concept of relative velocity, introduced in section 3.7 for motion along a straight line, can be easily extended to include motion in a plane or in three dimensions. Suppose that two objects A and B are moving with velocities $\mathbf{v_A}$ and $\mathbf{v_B}$ (each with respect to some common frame of reference, say ground.). Then, velocity of object A **relative to that of B is** :

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B \quad (4.35a)$$

and similarly, the velocity of object B *relative to that of* A is :

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$$

Therefore, $$\mathbf{v}_{AB} = -\mathbf{v}_{BA} \quad (4.35b)$$

and, $$|\mathbf{v}_{AB}| = |\mathbf{v}_{BA}| \quad (4.35c)$$

Example 4.6 Rain is falling vertically with a speed of 35 m s⁻¹. A woman rides a bicycle with a speed of 12 m s⁻¹ in east to west direction. What is the direction in which she should hold her umbrella ?

Answer In Fig. 4.16 $\mathbf{v}_r$ represents the velocity of rain and $\mathbf{v}_b$, the velocity of the bicycle, the woman is riding. Both these velocities are with respect to the ground. Since the woman is riding a bicycle, the velocity of rain as experienced by

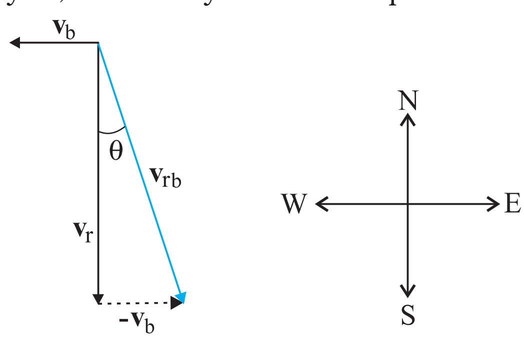


Fig. 4.16

her is the velocity of rain relative to the velocity of the bicycle she is riding. That is $\mathbf{v}_{rb} = \mathbf{v}_r - \mathbf{v}_b$

This relative velocity vector as shown in Fig. 4.16 makes an angle θ with the vertical. It is given by

$$\tan\theta = \frac{v_b}{v_r} = \frac{12}{35} = 0.343$$

Or, $\theta \cong 19^\circ$

Therefore, the woman should hold her umbrella at an angle of about 19° with the vertical towards the west.

Note carefully the difference between this Example and the Example 4.1. In Example 4.1, the boy experiences the resultant (vector sum) of two velocities while in this example, the woman experiences the velocity of rain relative to the bicycle (the vector difference of the two velocities). ◀

4.10 PROJECTILE MOTION

As an application of the ideas developed in the previous sections, we consider the motion of a projectile. An object that is in flight after being thrown or projected is called a **projectile**. Such a projectile might be a football, a cricket ball, a baseball or any other object. The motion of a projectile may be thought of as the result of two separate, simultaneously occurring components of motions. One component is along a horizontal direction without any acceleration and the other along the vertical direction with constant acceleration due to the force of gravity. It was Galileo who first stated this independency of the horizontal and the vertical components of projectile motion in his **Dialogue on the great world systems** (1632).

In our discussion, we shall assume that the air resistance has negligible effect on the motion of the projectile. Suppose that the projectile is launched with velocity $\mathbf{v}_o$ that makes an angle θ_o with the x-axis as shown in Fig. 4.17.

After the object has been projected, the acceleration acting on it is that due to gravity which is directed vertically downward:

$$\mathbf{a} = -g\,\hat{\mathbf{j}}$$

Or, $\quad a_x = 0,\ a_y = -g \qquad (4.36)$

The components of initial velocity $\mathbf{v}_o$ are :

$$v_{ox} = v_o \cos\theta_o$$
$$v_{oy} = v_o \sin\theta_o \qquad (4.37)$$

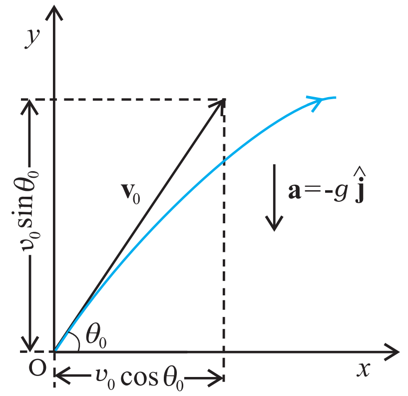


Fig 4.17 *Motion of an object projected with velocity $\mathbf{v}_o$ at angle θ_o.*

If we take the initial position to be the origin of the reference frame as shown in Fig. 4.17, we have :

$$x_o = 0,\ y_o = 0$$

Then, Eq.(4.34b) becomes :

$$x = v_{ox}\,t = (v_o \cos\theta_o)\,t$$

and $\quad y = (v_o \sin\theta_o)\,t - (\tfrac{1}{2})g\,t^2 \qquad (4.38)$

The components of velocity at time t can be obtained using Eq.(4.33b) :

$$v_x = v_{ox} = v_o \cos\theta_o$$
$$v_y = v_o \sin\theta_o - g\,t \qquad (4.39)$$

Equation (4.38) gives the x-, and y-coordinates of the position of a projectile at time t in terms of two parameters — initial speed v_o and projection angle θ_o. Notice that the choice of mutually perpendicular x-, and y-directions for the analysis of the projectile motion has resulted in a simplification. One of the components of velocity, i.e. x-component remains constant throughout the motion and only the y- component changes, like an object in free fall in vertical direction. This is shown graphically at few instants in Fig. 4.18. Note that at the point of maximum height, $v_y = 0$ and therefore,

$$\theta = \tan^{-1}\frac{v_y}{v_x} = 0$$

Equation of path of a projectile

What is the shape of the path followed by the projectile? This can be seen by eliminating the time between the expressions for x and y as given in Eq. (4.38). We obtain:

$$y = (\tan\theta_o)\,x - \frac{g}{2\,(v_o\cos\theta_o)^2}\,x^2 \qquad (4.40)$$

Now, since g, θ_o and v_o are constants, Eq. (4.40) is of the form $y = a\,x + b\,x^2$, in which a and b are constants. This is the equation of a parabola, i.e. the path of the projectile is a parabola (Fig. 4.18).

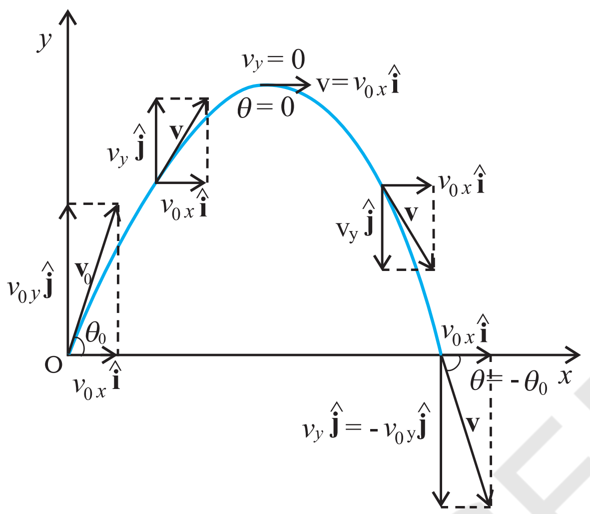


Fig. 4.18 *The path of a projectile is a parabola.*

Time of maximum height

How much time does the projectile take to reach the maximum height ? Let this time be denoted by t_m. Since at this point, $v_y = 0$, we have from Eq. (4.39):

$$v_y = v_o \sin\theta_o - g\,t_m = 0$$

Or, $\quad t_m = v_o \sin\theta_o / g \qquad (4.41a)$

The total time T_f during which the projectile is in flight can be obtained by putting $y = 0$ in Eq. (4.38). We get :

$$T_f = 2\,(v_o \sin\theta_o)/g \qquad (4.41b)$$

T_f is known as the **time of flight** of the projectile. We note that $T_f = 2\,t_m$, which is expected because of the symmetry of the parabolic path.

Maximum height of a projectile

The maximum height h_m reached by the projectile can be calculated by substituting $t = t_m$ in Eq. (4.38) :

$$y = h_m = (v_0\sin\theta_0)\left(\frac{v_0\sin\theta_0}{g}\right) - \frac{g}{2}\left(\frac{v_0\sin\theta_0}{g}\right)^2$$

Or, $$h_m = \frac{(v_0\sin\theta_0)^2}{2g} \qquad (4.42)$$

Horizontal range of a projectile

The horizontal distance travelled by a projectile from its initial position ($x = y = 0$) to the position where it passes $y = 0$ during its fall is called the **horizontal range**, R. It is the distance travelled during the time of flight T_f . Therefore, the range R is

$$R = (v_o\cos\theta_o)\,(T_f)$$
$$= (v_o\cos\theta_o)\,(2\,v_o\sin\theta_o)/g$$

Or, $$R = \frac{v_0^2\sin 2\theta_0}{g} \qquad (4.43a)$$

Equation (4.43a) shows that for a given projection velocity v_o , R is maximum when $\sin 2\theta_0$ is maximum, i.e., when $\theta_0 = 45^0$.

The maximum horizontal range is, therefore,

$$R_m = \frac{v_0^2}{g} \qquad (4.43b)$$

▶ ***Example 4.7*** Galileo, in his book **Two new sciences,** stated that "for elevations which exceed or fall short of 45° by equal amounts, the ranges are equal". Prove this statement.

Answer For a projectile launched with velocity $\mathbf{v}_o$ at an angle θ_o , the range is given by

$$R = \frac{v_0^2\sin 2\theta_0}{g}$$

Now, for angles, (45° + α) and (45° – α), $2\theta_o$ is (90° + 2α) and (90° – 2α) , respectively. The values of sin (90° + 2α) and sin (90° – 2α) are the same, equal to that of cos 2α. Therefore, ranges are equal for elevations which exceed or fall short of 45° by equal amounts α. ◀

▶ ***Example 4.8*** A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 m s^{-1}. Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed with which it hits the ground. (Take g = 9.8 m s^{-2}).

Answer We choose the origin of the x-,and y-axis at the edge of the cliff and t = 0 s at the instant the stone is thrown. Choose the positive direction of x-axis to be along the initial velocity and the positive direction of y-axis to be the vertically upward direction. The x-, and y-components of the motion can be treated independently. The equations of motion are :

$$x(t) = x_o + v_{ox}\, t$$
$$y(t) = y_o + v_{oy}\, t + (1/2)\, a_y\, t^2$$

Here, $x_o = y_o = 0$, $v_{oy} = 0$, $a_y = -g = -9.8 \text{ m s}^{-2}$, $v_{ox} = 15 \text{ m s}^{-1}$.

The stone hits the ground when $y(t) = -490$ m.

$$-490 \text{ m} = -(1/2)(9.8)\, t^2.$$

This gives $t = 10$ s.

The velocity components are $v_x = v_{ox}$ and $v_y = v_{oy} - g\,t$

so that when the stone hits the ground :

$$v_{ox} = 15 \text{ m s}^{-1}$$
$$v_{oy} = 0 - 9.8 \times 10 = -98 \text{ m s}^{-1}$$

Therefore, the speed of the stone is

$$\sqrt{v_x^2 + v_y^2} = \sqrt{15^2 + 98^2} = 99 \text{ m s}^{-1}$$

Example 4.9 A cricket ball is thrown at a speed of 28 m s^{-1} in a direction 30° above the horizontal. Calculate (a) the maximum height, (b) the time taken by the ball to return to the same level, and (c) the distance from the thrower to the point where the ball returns to the same level.

Answer (a) The maximum height is given by

$$h_m = \frac{(v_0 \sin\theta_o)^2}{2g} = \frac{(28 \sin 30°)^2}{2\ (9.8)} \text{ m}$$

$$= \frac{14 \times 14}{2 \times 9.8} = 10.0 \text{ m}$$

(b) The time taken to return to the same level is

$$T_f = (2\ v_o \sin\theta_o)/g = (2 \times 28 \times \sin 30°)/9.8$$
$$= 28/9.8 \text{ s} = 2.9 \text{ s}$$

(c) The distance from the thrower to the point where the ball returns to the same level is

$$R = \frac{(v_o^2 \sin 2\theta_o)}{g} = \frac{28 \times 28 \times \sin 60^o}{9.8} = 69 \text{ m}$$

Neglecting air resistance - what does the assumption really mean?

While treating the topic of projectile motion, we have stated that we assume that the air resistance has no effect on the motion of the projectile. You must understand what the statement really means. Friction, force due to viscosity, air resistance are all dissipative forces. In the presence of any of such forces opposing motion, any object will lose some part of its initial energy and consequently, momentum too. Thus, a projectile that traverses a parabolic path would certainly show deviation from its idealised trajectory in the presence of air resistance. It will not hit the ground with the same speed with which it was projected from it. In the absence of air resistance, the x-component of the velocity remains constant and it is only the y-component that undergoes a continuous change. However, in the presence of air resistance, both of these would get affected. That would mean that the range would be less than the one given by Eq. (4.43). Maximum height attained would also be less than that predicted by Eq. (4.42). Can you then, anticipate the change in the time of flight?

In order to avoid air resistance, we will have to perform the experiment in vacuum or under low pressure, which is not easy. When we use a phrase like 'neglect air resistance', we imply that the change in parameters such as range, height etc. is much smaller than their values without air resistance. The calculation without air resistance is much simpler than that with air resistance.

4.11 UNIFORM CIRCULAR MOTION

When an object follows a circular path at a constant speed, the motion of the object is called **uniform circular motion**. The word "uniform" refers to the speed, which is uniform (constant) throughout the motion. Suppose an object is moving with uniform speed v in a circle of radius R as shown in Fig. 4.19. Since the velocity of the object is changing continuously in direction, the object undergoes acceleration. Let us find the magnitude and the direction of this acceleration.

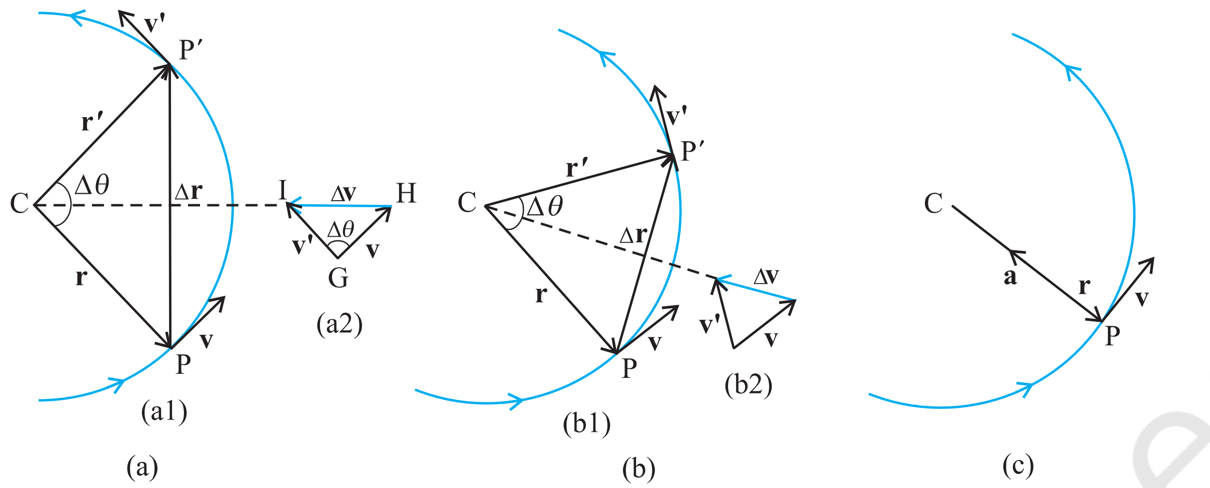


Fig. 4.19 *Velocity and acceleration of an object in uniform circular motion. The time interval Δt decreases from (a) to (c) where it is zero. The acceleration is directed, at each point of the path, towards the centre of the circle.*

Let **r** and **r**′ be the position vectors and **v** and **v**′ the velocities of the object when it is at point *P* and *P*′ as shown in Fig. 4.19(a). By definition, velocity at a point is along the tangent at that point in the direction of motion. The velocity vectors **v** and **v**′ are as shown in Fig. 4.19(a1). Δ**v** is obtained in Fig. 4.19 (a2) using the triangle law of vector addition. Since the path is circular, **v** is perpendicular to **r** and so is **v**′ to **r**′. Therefore, Δ**v** is perpendicular to Δ**r**. Since average acceleration is along Δ**v** $\left(\bar{\mathbf{a}} = \frac{\Delta \mathbf{v}}{\Delta t}\right)$, the average acceleration $\bar{\mathbf{a}}$ is perpendicular to Δ**r.** If we place Δ**v** on the line that bisects the angle between **r** and **r**′, we see that it is directed towards the centre of the circle. Figure 4.19(b) shows the same quantities for smaller time interval. Δ**v** and hence $\bar{\mathbf{a}}$ is again directed towards the centre. In Fig. 4.19(c), Δ*t*→ 0 and the average acceleration becomes the instantaneous acceleration. It is directed towards the centre*. Thus, we find that the acceleration of an object in uniform circular motion is always directed towards the centre of the circle. Let us now find the magnitude of the acceleration.

The magnitude of **a** is, by definition, given by

$$|\mathbf{a}| = \lim_{\Delta t \to 0} \frac{|\Delta \mathbf{v}|}{\Delta t}$$

Let the angle between position vectors **r** and **r**′ be Δ*θ*. Since the velocity vectors **v** and **v**′ are always perpendicular to the position vectors, the angle between them is also Δ*θ* . Therefore, the triangle CPP′ formed by the position vectors and the triangle GHI formed by the velocity vectors **v**, **v**′ and Δ**v** are similar (Fig. 4.19a). Therefore, the ratio of the base-length to side-length for one of the triangles is equal to that of the other triangle. That is :

$$\frac{|\Delta \mathbf{v}|}{v} = \frac{|\Delta \mathbf{r}|}{R}$$

Or, $$|\Delta \mathbf{v}| = v \frac{|\Delta \mathbf{r}|}{R}$$

Therefore,

$$|\mathbf{a}| = \lim_{\Delta t \to 0} \frac{|\Delta \mathbf{v}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{v|\Delta \mathbf{r}|}{R\Delta t} = \frac{v}{R} \lim_{\Delta t \to 0} \frac{|\Delta \mathbf{r}|}{\Delta t}$$

If Δt is small, Δθ will also be small and then arc *PP*′ can be approximately taken to be |Δ**r**|:

$$|\Delta \mathbf{r}| \cong v\Delta t$$

$$\frac{|\Delta \mathbf{r}|}{\Delta t} \cong v$$

Or, $$\lim_{\Delta t \to 0} \frac{|\Delta \mathbf{r}|}{\Delta t} = v$$

Therefore, the centripetal acceleration a_c is :

* *In the limit Δt→0, Δ**r** becomes perpendicular to **r**. In this limit Δ**v**→**0** and is consequently also perpendicular to **V**. Therefore, the acceleration is directed towards the centre, at each point of the circular path.*

$$a_c = \left(\frac{v}{R}\right) v = v^2/R \quad (4.44)$$

Thus, the acceleration of an object moving with speed v in a circle of radius R has a magnitude v^2/R and is always **directed towards the centre**. This is why this acceleration is called **centripetal acceleration** (a term proposed by Newton). A thorough analysis of centripetal acceleration was first published in 1673 by the Dutch scientist Christiaan Huygens (1629-1695) but it was probably known to Newton also some years earlier. "Centripetal" comes from a Greek term which means 'centre-seeking'. Since v and R are constant, the magnitude of the centripetal acceleration is also constant. However, the direction changes — pointing always towards the centre. Therefore, a centripetal acceleration is not a constant vector.

We have another way of describing the velocity and the acceleration of an object in uniform circular motion. As the object moves from P to P′ in time Δt (= t' – t), the line CP (Fig. 4.19) turns through an angle $\Delta\theta$ as shown in the figure. $\Delta\theta$ is called angular distance. We define the angular speed ω (Greek letter omega) as the time rate of change of angular displacement :

$$\omega = \frac{\Delta\theta}{\Delta t} \quad (4.45)$$

Now, if the distance travelled by the object during the time Δt is Δs, i.e. PP' is Δs, then :

$$v = \frac{\Delta s}{\Delta t}$$

but $\Delta s = R\,\Delta\theta$. Therefore :

$$v = R\frac{\Delta\theta}{\Delta t} = R\,\omega$$

$$v = R\,\omega \quad (4.46)$$

We can express centripetal acceleration a_c in terms of angular speed :

$$a_c = \frac{v^2}{R} = \frac{\omega^2 R^2}{R} = \omega^2 R$$

$$a_c = \omega^2 R \quad (4.47)$$

The time taken by an object to make one revolution is known as its time period T and the number of revolution made in one second is called its frequency ν (=$1/T$). However, during this time the distance moved by the object is $s = 2\pi R$.

Therefore, $v = 2\pi R/T = 2\pi R\nu$ (4.48)

In terms of frequency ν, we have

$\omega = 2\pi\nu$

$v = 2\pi R\nu$

$a_c = 4\pi^2 \nu^2 R$ (4.49)

> **Example 4.10** An insect trapped in a circular groove of radius 12 cm moves along the groove steadily and completes 7 revolutions in 100 s. (a) What is the angular speed, and the linear speed of the motion? (b) Is the acceleration vector a constant vector ? What is its magnitude ?

Answer This is an example of uniform circular motion. Here R = 12 cm. The angular speed ω is given by

$\omega = 2\pi/T = 2\pi \times 7/100 = 0.44$ rad/s

The linear speed v is :

$v = \omega R = 0.44\ s^{-1} \times 12\ cm = 5.3\ cm\ s^{-1}$

The direction of velocity $\boldsymbol{v}$ is along the tangent to the circle at every point. The acceleration is directed towards the centre of the circle. Since this direction changes continuously, acceleration here is *not* a constant vector. However, the magnitude of acceleration is constant:

$a = \omega^2 R = (0.44\ s^{-1})^2\ (12\ cm)$

$= 2.3\ cm\ s^{-2}$ ◀