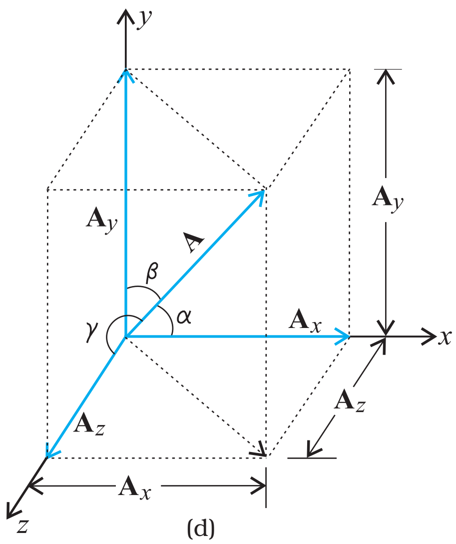
So far we have considered a vector lying in an x-y plane. The same procedure can be used to resolve a general vector **A** into three components along x-, y-, and z-axes in three dimensions. If α, β, and γ are the angles* between **A** and the x-, y-, and z-axes, respectively [Fig. 4.9(d)], we have



(d)

Fig. 4.9 (d) *A vector **A** resolved into components along x-, y-, and z-axes*

$$A_x = A\cos\alpha,\ A_y = A\cos\beta,\ A_z = A\cos\gamma \quad (4.16a)$$

In general, we have

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \quad (4.16b)$$

The magnitude of vector **A** is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (4.16c)$$

A position vector **r** can be expressed as

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \quad (4.17)$$

where x, y, and z are the components of **r** along x-, y-, z-axes, respectively.

4.6 VECTOR ADDITION – ANALYTICAL METHOD

Although the graphical method of adding vectors helps us in visualising the vectors and the resultant vector, it is sometimes tedious and has limited accuracy. It is much easier to add vectors by combining their respective components. Consider two vectors **A** and **B** in x-y plane with components A_x, A_y and B_x, B_y :

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} \quad (4.18)$$

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}$$

Let **R** be their sum. We have

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

$$= \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}\right) \quad (4.19a)$$

Since vectors obey the commutative and associative laws, we can arrange and regroup the vectors in Eq. (4.19a) as convenient to us :

$$\mathbf{R} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}} \quad (4.19b)$$

$$\text{Since}\ \mathbf{R} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} \quad (4.20)$$

$$\text{we have,}\ R_x = A_x + B_x,\ R_y = A_y + B_y \quad (4.21)$$

Thus, each component of the resultant vector **R** is the sum of the corresponding components of **A** and **B**.

In three dimensions, we have

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} + R_z\hat{\mathbf{k}}$$

with

$$R_x = A_x + B_x$$
$$R_y = A_y + B_y$$
$$R_z = A_z + B_z \quad (4.22)$$

This method can be extended to addition and subtraction of any number of vectors. For example, if vectors **a**, **b** and **c** are given as

$$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$$

$$\mathbf{b} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$$

$$\mathbf{c} = c_x\hat{\mathbf{i}} + c_y\hat{\mathbf{j}} + c_z\hat{\mathbf{k}} \quad (4.23a)$$

then, a vector **T** = **a** + **b** – **c** has components :

$$T_x = a_x + b_x - c_x$$
$$T_y = a_y + b_y - c_y \quad (4.23b)$$
$$T_z = a_z + b_z - c_z.$$

▶ ***Example 4.2*** Find the magnitude and direction of the resultant of two vectors **A** and **B** in terms of their magnitudes and angle θ between them.

* *Note that angles α, β, and γ are angles in space. They are between pairs of lines, which are not coplanar.*

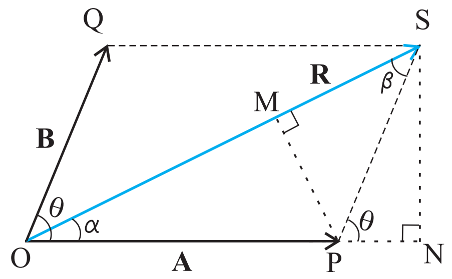


Fig. 4.10

Answer Let **OP** and **OQ** represent the two vectors **A** and **B** making an angle θ (Fig. 4.10). Then, using the parallelogram method of vector addition, **OS** represents the resultant vector **R** :

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

SN is normal to *OP* and *PM* is normal to *OS*.

From the geometry of the figure,

$$OS^2 = ON^2 + SN^2$$

but $\quad ON = OP + PN = A + B\cos\theta$

$SN = B\sin\theta$

$$OS^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$

or, $\quad R^2 = A^2 + B^2 + 2AB\cos\theta$

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta} \tag{4.24a}$$

In Δ OSN, $\quad SN = OS\sin\alpha = R\sin\alpha$, and

in Δ PSN, $\quad SN = PS\sin\theta = B\sin\theta$

Therefore, $\quad R\sin\alpha = B\sin\theta$

or, $$\frac{R}{\sin\theta} = \frac{B}{\sin\alpha} \tag{4.24b}$$

Similarly,

$$\text{PM} = A\sin\alpha = B\sin\beta$$

or, $$\frac{A}{\sin\beta} = \frac{B}{\sin\alpha} \tag{4.24c}$$

Combining Eqs. (4.24b) and (4.24c), we get

$$\frac{R}{\sin\theta} = \frac{A}{\sin\beta} = \frac{B}{\sin\alpha} \tag{4.24d}$$

Using Eq. (4.24d), we get:

$$\sin\alpha = \frac{B}{R}\sin\theta \tag{4.24e}$$

where *R* is given by Eq. (4.24a).

or, $$\tan\alpha = \frac{SN}{OP + PN} = \frac{B\sin\theta}{A + B\cos\theta} \tag{4.24f}$$

Equation (4.24a) gives the magnitude of the resultant and Eqs. (4.24e) and (4.24f) its direction. Equation (4.24a) is known as the **Law of cosines** and Eq. (4.24d) as the **Law of sines**. ◀

▶ ***Example 4.3*** A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.

Answer The vector $\mathbf{v_b}$ representing the velocity of the motorboat and the vector $\mathbf{v_c}$ representing the water current are shown in Fig. 4.11 in directions specified by the problem. Using the parallelogram method of addition, the resultant **R** is obtained in the direction shown in the figure.

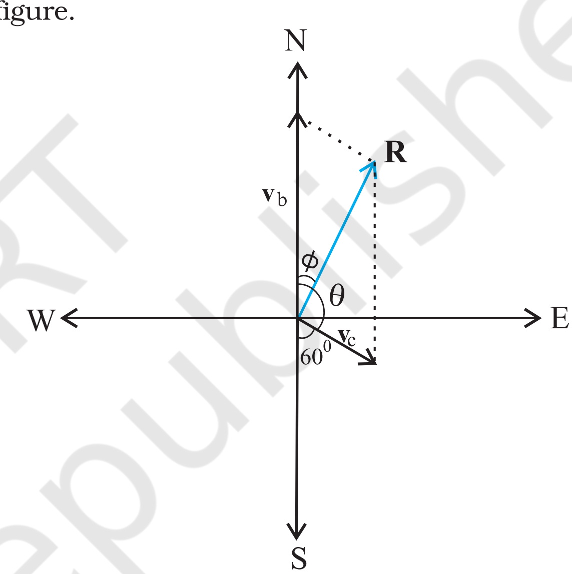


Fig. 4.11

We can obtain the magnitude of **R** using the Law of cosine :

$$R = \sqrt{v_b^2 + v_c^2 + 2v_b v_c \cos 120^\circ}$$

$$= \sqrt{25^2 + 10^2 + 2\times 25\times 10(-1/2)} \cong 22 \text{ km/h}$$

To obtain the direction, we apply the Law of sines

$$\frac{R}{\sin\theta} = \frac{v_c}{\sin\phi} \quad \text{or, } \sin\phi = \frac{v_c}{R}\sin\theta$$

$$= \frac{10\times\sin 120^\circ}{21.8} = \frac{10\sqrt{3}}{2\times 21.8} \cong 0.397$$

$$\phi \cong 23.4^\circ$$ ◀

4.7 MOTION IN A PLANE

In this section we shall see how to describe motion in two dimensions using vectors.