Question 1

The general solution of the differential equation $(y^2 - x^3)dx - xydy = 0$ $(x \neq 0)$ is (where, C is a constant of integration) **(2019 Main, 12 April II)**

(a) $y^2 - 2x^2 + Cx^3 = 0$ (b) $y^2 + 2x^3 + Cx^2 = 0$
(c) $y^2 + 2x^2 + Cx^3 = 0$ (d) $y^2 - 2x^3 + Cx^2 = 0$

Answer: (b)
Solution:

Given differential equation is

$$(y^2 - x^3)\,dx - xy\,dy = 0,\ (x \neq 0)$$

$$\Rightarrow \quad xy\frac{dy}{dx} - y^2 = -x^3$$

Now, put $y^2 = t \Rightarrow 2y\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow y\frac{dy}{dx} = \frac{1}{2}\frac{dt}{dx}$

$$\therefore \quad \frac{x}{2}\frac{dt}{dx} - t = -x^3$$

$$\Rightarrow \quad \frac{dt}{dx} - \frac{2}{x}t = -2x^2$$

which is the linear differential equation of the form

$$\frac{dt}{dx} + Pt = Q.$$

Here, $P = -\frac{2}{x}$ and $Q = -2x^2$.

Now, $\text{IF} = e^{-\int \frac{2}{x}dx} = \frac{1}{x^2}$

$\because$ Solution of the linear differential equation is

$(\text{IF})\,t = \int Q(\text{IF})dx + \lambda$ [where λ is integrating constant]

$$\therefore \quad t\left(\frac{1}{x^2}\right) = -2\int\left(x^2 \times \frac{1}{x^2}\right)dx + \lambda$$

$$\Rightarrow \quad \frac{t}{x^2} = -2x + \lambda$$

$$\Rightarrow \quad \frac{y^2}{x^2} + 2x = \lambda \qquad [\because t = y^2]$$

$$\Rightarrow \quad y^2 + 2x^3 - \lambda x^2 = 0$$

$$\text{or} \quad y^2 + 2x^3 + Cx^2 = 0 \qquad [\text{let } C = -\lambda]$$

Question 2

Consider the differential equation, $y^2dx + \left(x - \frac{1}{y}\right)dy = 0$.

If value of y is 1 when $x = 1$, then the value of x for which $y = 2$, is **(2019 Main, 12 April I)**

(a) $\frac{5}{2} + \frac{1}{\sqrt{e}}$ (b) $\frac{3}{2} - \frac{1}{\sqrt{e}}$ (c) $\frac{1}{2} + \frac{1}{\sqrt{e}}$ (d) $\frac{3}{2} - \sqrt{e}$

Answer: (b)
Solution:

Given differential equation is

$$y^2dx + \left(x - \frac{1}{y}\right)dy = 0$$

$\Rightarrow \frac{dx}{dy} + \frac{1}{y^2}x = \frac{1}{y^3}$, which is the linear differential equation of the form $\frac{dx}{dy} + Px = Q$.

Here, $P = \frac{1}{y^2}$ and $Q = \frac{1}{y^3}$

Now, $\text{IF} = e^{\int \frac{1}{y^2}dy} = e^{-\frac{1}{y}}$

$\therefore$ The solution of linear differential equation is

$x \cdot (\text{IF}) \int Q(\text{IF})dy + C$

$\therefore$ The solution of linear differential equation is

$x \cdot (\text{IF}) \int Q(\text{IF})dy + C$

$$\Rightarrow x e^{-1/y} = \int \frac{1}{y^3} e^{-1/y}dy + C$$

$$\therefore x e^{-1/y} = \int (-t)\, e^t dt + C \qquad \left[\because \text{let} -\frac{1}{y} = t \Rightarrow +\frac{1}{y^2}dy = dt\right]$$

$$= -te^t + \int e^t\,dt + C \qquad [\text{integration by parts}]$$

$$= -te^t + e^t + C$$

$$\Rightarrow x e^{-1/y} = \frac{1}{y}e^{-1/y} + e^{-1/y} + C \qquad \ldots(\text{i})$$

Now, at $y = 1$, the value of $x = 1$, so

$$1 \cdot e^{-1} = e^{-1} + e^{-1} + C \Rightarrow C = -\frac{1}{e}$$

On putting the value of C, in Eq. (i), we get

$$x = \frac{1}{y} + 1 - \frac{e^{1/y}}{e}$$

So, at $y = 2$, the value of $x = \frac{1}{2} + 1 - \frac{e^{1/2}}{e} = \frac{3}{2} - \frac{1}{\sqrt{e}}$

Question 3

Let $y = y(x)$ be the solution of the differential equation, $\frac{dy}{dx} + y\tan x = 2x + x^2\tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $y(0) = 1$. Then **(2019 Main, 10 April II)**

(a) $y'\left(\frac{\pi}{4}\right) - y'\left(-\frac{\pi}{4}\right) = \pi - \sqrt{2}$ (b) $y'\left(\frac{\pi}{4}\right) + y'\left(-\frac{\pi}{4}\right) = -\sqrt{2}$

(c) $y\left(\frac{\pi}{4}\right) + y\left(-\frac{\pi}{4}\right) = \frac{\pi^2}{2} + 2$ (d) $y\left(\frac{\pi}{4}\right) - y\left(-\frac{\pi}{4}\right) = \sqrt{2}$

Answer: (a)
Solution:

Given differential equation is

$\frac{dy}{dx} + y\tan x = 2x + x^2\tan x$, which is linear differential equation in the form of $\frac{dy}{dx} + Py = Q$.

Here, $P = \tan x$ and $Q = 2x + x^2 \tan x$

$\therefore \text{IF} = e^{\int \tan x\, dx} = e^{\log_e(\sec x)} = \sec x$

Now, solution of linear differential equation is given as

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C$$

$$\therefore y(\sec x) = \int (2x + x^2 \tan x) \sec x\, dx + C$$

$$= \int (2x\sec x)\, dx + \int x^2 \sec x \tan x\, dx + C$$

$$\because \int x^2 \sec x \tan x\, dx = x^2 \sec x - \int (2x \sec x)\, dx$$

Therefore, solution is

$$y\sec x = 2\int x\sec x\, dx + x^2 \sec x - 2\int x\sec x\, dx + C$$

$\Rightarrow y\sec x = x^2 \sec x + C$ ...(i)

$\because y(0) = 1 \Rightarrow 1(1) = 0(1) + C \Rightarrow C = 1$

Now, $y = x^2 + \cos x$ [from Eq. (i)]

and $y' = 2x - \sin x$

According to options,

$$y'\left(\frac{\pi}{4}\right) - y'\left(\frac{-\pi}{4}\right) = \left(2\left(\frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}\right) - \left(2\left(-\frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}\right) = \pi - \sqrt{2}$$

and $y'\left(\frac{\pi}{4}\right) + y'\left(-\frac{\pi}{4}\right) = \left(2\left(\frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}\right) + \left(2\left(-\frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}\right) = 0$

and $y\left(\frac{\pi}{4}\right) + y\left(-\frac{\pi}{4}\right) = \frac{\pi^2}{16} + \frac{1}{\sqrt{2}} + \frac{\pi^2}{16} + \frac{1}{\sqrt{2}} = \frac{\pi^2}{4} + \sqrt{2}$

and $y\left(\frac{\pi}{4}\right) - y\left(-\frac{\pi}{4}\right) = \frac{\pi^2}{16} + \frac{1}{\sqrt{2}} - \frac{\pi^2}{16} - \frac{1}{\sqrt{2}} = 0$

Question 4

If $y = y(x)$ is the solution of the differential equation $\frac{dy}{dx} = (\tan x - y)\sec^2 x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, such that $y(0) = 0$, then $y\left(-\frac{\pi}{4}\right)$ is equal to **(2019 Main, 10 April I)**

(a) $\frac{1}{e} - 2$ (b) $\frac{1}{2} - e$ (c) $2 + \frac{1}{e}$ (d) $e - 2$

Answer: (d)

Solution:

Given differential equation

$$\frac{dy}{dx} = (\tan x - y)\sec^2 x$$

$$\Rightarrow \frac{dy}{dx} + (\sec^2 x) y = \sec^2 x \tan x,$$

which is linear differential equation of the form

$$\frac{dy}{dx} + Py = Q,$$

where $P = \sec^2 x$ and $Q = \sec^2 x \tan x$

$$\text{IF} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

So, solution of given differential equation is

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C$$

$$y(e^{\tan x}) = \int e^{\tan x} \cdot \sec^2 x \tan x\, dx + C$$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$y e^{\tan x} = \int e^t \cdot t\, dt + C = te^t - \int e^t\, dt + C$$

[using integration by parts method]

$$= e^t (t - 1) + C$$

$\Rightarrow y \cdot e^{\tan x} = e^{\tan x}(\tan x - 1) + C$ $[\because t = \tan x]$

$\because y(0) = 0$

$\Rightarrow 0 = 1(0 - 1) + C \Rightarrow C = 1$

$\therefore y \cdot e^{\tan x} = e^{\tan x}(\tan x - 1) + 1$

Now, at $x = -\frac{\pi}{4}$

$$ye^{-1} = e^{-1}(-1 - 1) + 1$$

$\Rightarrow ye^{-1} = -2e^{-1} + 1 \Rightarrow y = e - 2$

Question 5

If $\cos x \frac{dy}{dx} - y \sin x = 6x$, $\left(0 < x < \frac{\pi}{2}\right)$ and $y\left(\frac{\pi}{3}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to **(2019 Main, 9 April II)**

(a) $\frac{\pi^2}{2\sqrt{3}}$ (b) $-\frac{\pi^2}{2\sqrt{3}}$ (c) $-\frac{\pi^2}{4\sqrt{3}}$ (d) $-\frac{\pi^2}{2}$

Answer: (b)

Solution:

Key Idea (i) First convert the given differential equation into linear differential equation of the form $\frac{dy}{dx} + Py = Q$

(ii) Find IF

(iii) Apply formula, $y(IF) = \int Q(IF) dx + C$

Given differential equation

$$\cos x \frac{dy}{dx} - (\sin x) y = 6x$$

$\Rightarrow \frac{dy}{dx} - (\tan x) y = \frac{6x}{\cos x}$, which is the linear differential equation of the form

$$\frac{dy}{dx} + Px = Q,$$

where $P = -\tan x$ and $Q = \frac{6x}{\cos x}$

So, $\text{IF} = e^{-\int \tan x\, dx} = e^{-\log(\sec x)} = \cos x$

$\therefore$ Required solution of differential equation is

$$y(\cos x) = \int (6x) \frac{\cos x}{\cos x} dx + C = 6\frac{x^2}{2} + C = 3x^2 + C$$

Given, $y\left(\frac{\pi}{3}\right) = 0$

So, $0 = 3\left(\frac{\pi}{3}\right)^2 + C \Rightarrow C = -\frac{\pi^2}{3}$

$\therefore y(\cos x) = 3x^2 - \frac{\pi^2}{3}$

Now, at $x = \frac{\pi}{6}$

$$y\left(\frac{\sqrt{3}}{2}\right) = 3\frac{\pi^2}{36} - \frac{\pi^2}{3} = -\frac{\pi^2}{4} \Rightarrow y = -\frac{\pi^2}{2\sqrt{3}}$$

Question 6

The solution of the differential equation $x\frac{dy}{dx}+2y=x^2(x\neq 0)$ with $y(1)=1$, is **(2019 Main, 9 April I)**

(a) $y=\frac{x^2}{4}+\frac{3}{4x^2}$ (b) $y=\frac{x^3}{5}+\frac{1}{5x^2}$

(c) $y=\frac{3}{4}x^2+\frac{1}{4x^2}$ (d) $y=\frac{4}{5}x^3+\frac{1}{5x^2}$

Answer: (a)
Solution:

Given differential equation is

$$x\frac{dy}{dx}+2y=x^2,\ (x\neq 0)$$

$$\Rightarrow \quad \frac{dy}{dx}+\left(\frac{2}{x}\right)y=x,$$

which is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q$$

Here, $P=\frac{2}{x}$ and $Q=x$

$$\therefore \quad \text{IF}=e^{\int\frac{2}{x}dx}=e^{2\log x}=x^2$$

Since, solution of the given differential equation is

$$y\times\text{IF}=\int(Q\times\text{IF})\,dx+C$$

$$\therefore\ y(x^2)=\int(x\times x^2)\,dx+C \Rightarrow yx^2=\frac{x^4}{4}+C$$

$$\because\ y(1)=1,\text{ so } 1=\frac{1}{4}+C \quad\Rightarrow C=\frac{3}{4}$$

$$\therefore\quad yx^2=\frac{x^4}{4}+\frac{3}{4}\Rightarrow y=\frac{x^2}{4}+\frac{3}{4x^2}$$

Question 7

Let $y=y(x)$ be the solution of the differential equation, $(x^2+1)^2\frac{dy}{dx}+2x(x^2+1)y=1$ such that $y(0)=0$. If $\sqrt{a}\ y(1)=\frac{\pi}{32}$, then the value of '$a$' is **(2019 Main, 8 April I)**

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) $\frac{1}{16}$

Answer: (d)
Solution:

Given differential equation is

$$(x^2+1)^2\frac{dy}{dx}+2x(x^2+1)y=1$$

$$\Rightarrow\quad \frac{dy}{dx}+\left(\frac{2x}{1+x^2}\right)y=\frac{1}{(1+x^2)^2}$$

[dividing each term by $(1+x^2)^2$] ...(i)

This is a linear differential equation of the form

$$\frac{dy}{dx}+P\cdot y=Q$$

Here, $P=\frac{2x}{(1+x^2)}$ and $Q=\frac{1}{(1+x^2)^2}$

$$\therefore \text{Integrating Factor (IF)}=e^{\int\frac{2x}{1+x^2}dx}$$

$$=e^{\ln(1+x^2)}=(1+x^2)$$

and required solution of differential Eq. (i) is given by

$$y\cdot(\text{IF})=\int Q(\text{IF})dx+C$$

$$\Rightarrow y(1+x^2)=\int\frac{1}{(1+x^2)^2}(1+x^2)dx+C$$

$$\Rightarrow y(1+x^2)=\int\frac{dx}{1+x^2}+C$$

$$\Rightarrow y(1+x^2)=\tan^{-1}(x)+C$$

$$\because\quad y(0)=0$$

$$\therefore\quad C=0$$

$$\therefore\quad y(1+x^2)=\tan^{-1}x \qquad [\because C=0]$$

$$\Rightarrow\quad y=\frac{\tan^{-1}x}{1+x^2}$$

$$\Rightarrow\quad \sqrt{a}y=\sqrt{a}\left(\frac{\tan^{-1}x}{1+x^2}\right)$$

[multiplying both sides by $\sqrt{a}$]

Now, at $x=1$

$$\sqrt{a}\ y\ (1)=\sqrt{a}\left(\frac{\tan^{-1}(1)}{1+1}\right)=\sqrt{a}\frac{\frac{\pi}{4}}{2}=\frac{\sqrt{a}\pi}{8}=\frac{\pi}{32}\text{ (given)}$$

$$\therefore\quad \sqrt{a}=\frac{1}{4}\Rightarrow a=\frac{1}{16}$$

Question 8

If a curve passes through the point $(1,-2)$ and has slope of the tangent at any point (x, y) on it as $\frac{x^2-2y}{x}$, then the curve also passes through the point **(2019 Main, 12 Jan II)**

(a) $(\sqrt{3},0)$ (b) $(-1,2)$

(c) $(-\sqrt{2},1)$ (d) $(3,0)$

Answer: (a)
Solution:

We know that, slope of the tangent at any point (x, y) on the curve is

$$\frac{dy}{dx}=\frac{x^2-2y}{x} \qquad \text{(given)}$$

$$\Rightarrow\quad \frac{dy}{dx}+\frac{2}{x}y=x \qquad \text{...(i)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx}+P(x)\cdot y=Q(x),$$

where $P(x)=\frac{2}{x}$ and $Q(x)=x$

Now, integrating factor

$$(\text{IF})=e^{\int P(x)dx}=e^{\int\frac{2}{x}dx}=e^{2\log_e x}$$

$$= e^{\log_e x^2} \quad [\because m \log a = \log a^m]$$
$$= x^2 \quad [\because e^{\log_e f(x)} = f(x)]$$

and the solution of differential Eq. (i) is

$$y(\text{IF}) = \int Q(x)(\text{IF})dx + C \Rightarrow y(x^2) = \int x \cdot x^2\, dx + C$$

$$\Rightarrow \quad yx^2 = \frac{x^4}{4} + C \quad \ldots\text{(ii)}$$

$\because$ The curve (ii) passes through the point $(1, -2)$, therefore

$$-2 = \frac{1}{4} + C \Rightarrow C = -\frac{9}{4}$$

$\therefore$ Equation of required curve is $4yx^2 = x^4 - 9$.

Now, checking all the option, we get only $(\sqrt{3}, 0)$ satisfy the above equation.

Question 9

Let $y = y(x)$ be the solution of the differential equation, $x\frac{dy}{dx} + y = x\log_e x, (x > 1)$. If $2y(2) = \log_e 4 - 1$, then $y(e)$ is equal to **(2019 Main, 12 Jan I)**

(a) $-\frac{e}{2}$ (b) $-\frac{e^2}{2}$ (c) $\frac{e}{4}$ (d) $\frac{e^2}{4}$

Answer: (c)

Solution:

Given differential equation is

$$x\frac{dy}{dx} + y = x\log_e x, (x > 1)$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{x}y = \log_e x \quad \ldots\text{(i)}$$

Which is a linear differential equation.

So, if $= e^{\int \frac{1}{x} dx} = e^{\log_e x} = x$

Now, solution of differential Eq. (i), is

$$y \times x = \int (\log_e x)\, x\, dx + C$$

$$\Rightarrow \quad yx = \frac{x^2}{2}\log_e x - \int \frac{x^2}{2} \times \frac{1}{x}\, dx + C$$

[using integration by parts]

$$\Rightarrow \quad yx = \frac{x^2}{2}\log_e x - \frac{x^2}{4} + C \quad \ldots\text{(ii)}$$

Given that, $2y(2) = \log_e 4 - 1$ ... (iii)

On substituting, $x = 2$, in Eq. (ii), we get

$$2y(2) = \frac{4}{2}\log_e 2 - \frac{4}{4} + C,$$

[where, $y(2)$ represents value of y at $x = 2$]

$$\Rightarrow \quad 2y(2) = \log_e 4 - 1 + C \quad \ldots\text{(iv)}$$

$[\because m \log a = \log a^m]$

From Eqs. (iii) and (iv), we get

$$C = 0$$

So, required solution is

$$yx = \frac{x^2}{2}\log_e x - \frac{x^2}{4}$$

Now, at $\quad x = e,\ ey(e) = \frac{e^2}{2}\log_e e - \frac{e^2}{4}$

[where, $y(e)$ represents value of y at $x = e$]

$$\Rightarrow \quad y(e) = \frac{e}{4} \quad [\because \log_e e = 1].$$

Question 10

If $y(x)$ is the solution of the differential equation

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x},\ x > 0,$$

where $y(1) = \frac{1}{2}e^{-2}$, then **(2019 Main, 11 Jan I)**

(a) $y(x)$ is decreasing in $\left(\frac{1}{2}, 1\right)$

(b) $y(x)$ is decreasing in $(0, 1)$

(c) $y(\log_e 2) = \log_e 4$

(d) $y(\log_e 2) = \frac{\log_e 2}{4}$

Answer: (a)

Solution:

We have, $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$

which is of the form $\frac{dy}{dx} + Py = Q$, where

$$P = \frac{2x+1}{x} \text{ and } Q = e^{-2x}$$

Now, IF $= e^{\int P dx} = e^{\int\left(\frac{1+2x}{x}\right)dx} = e^{\int\left(\frac{1}{x}+2\right)dx}$

$$= e^{\ln x + 2x} = e^{\ln x} \cdot e^{2x} = x\, e^{2x}$$

and the solution of the given equation is

$$y \cdot (\text{IF}) = \int (\text{IF}) Q\, dx + C$$

$$\Rightarrow \quad y(xe^{2x}) = \int (x e^{2x} \cdot e^{-2x})\, dx + C$$

$$= \int x\, dx + C = \frac{x^2}{2} + C \quad \ldots\text{(i)}$$

Since, $y = \frac{1}{2}e^{-2}$ when $x = 1$

$\therefore \quad \frac{1}{2}e^{-2} \cdot e^2 = \frac{1}{2} + C \Rightarrow C = 0$ (using Eq. (i))

$\therefore \quad y\,(xe^{2x}) = \frac{x^2}{2} \quad \Rightarrow y = \frac{x}{2}e^{-2x}$

Now, $\frac{dy}{dx} = \frac{1}{2}e^{-2x} + \frac{x}{2}e^{-2x}\,(-2) = e^{-2x}\left\{\frac{1}{2} - x\right\} < 0,$

if $\frac{1}{2} < x < 1$ [by using product rule of derivative]

and $y(\log_e 2) = \frac{\log_e 2}{2}\, e^{-2\log_e 2} = \frac{1}{2}\log_e 2\, e^{\log_e 2^{-2}}$

$$= \frac{1}{2} \cdot \log_e 2 \cdot 2^{-2} = \frac{1}{8}\log_e 2$$

Question 11

Let f be a differentiable function such that $f'(x) = 7 - \frac{3}{4}\frac{f(x)}{x}$, $(x > 0)$ and $f(1) \neq 4$. Then, $\lim_{x \to 0^+} x f\left(\frac{1}{x}\right)$

(2019 Main, 10 Jan II)

(a) does not exist (b) exists and equals $\frac{4}{7}$

(c) exists and equals 0 (d) exists and equals 4

Answer: (d)

Solution:

Given, $f'(x) = 7 - \frac{3}{4}\frac{f(x)}{x}$, $(x > 0)$

On putting $f(x) = y$ and $f'(x) = \frac{dy}{dx}$, then we get

$$\frac{dy}{dx} = 7 - \frac{3}{4}\frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{3}{4x} y = 7 \quad \ldots(i)$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{3}{4x}$ and $Q = 7$.

Now, integrating factor (IF) $= e^{\int \frac{3}{4x} dx}$

$$= e^{\frac{3}{4}\log x} = e^{\log x^{3/4}} = x^{3/4}$$

and solution of differential Eq. (i) is given by

$$y(\text{IF}) = \int (Q \cdot (\text{IF}))dx + C$$

$$yx^{3/4} = \int 7x^{3/4} dx + C$$

$$\Rightarrow \quad y\, x^{3/4} = 7\frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1} + C$$

$$\Rightarrow \quad y\, x^{3/4} = 4x^{\frac{7}{4}} + C$$

$$\Rightarrow \quad y = 4x + C\, x^{-3/4}$$

So, $y = f(x) = 4x + C \cdot x^{-3/4}$

Now, $f\left(\frac{1}{x}\right) = \frac{4}{x} + C \cdot x^{3/4}$

$$\therefore \lim_{x \to 0^+} x f\left(\frac{1}{x}\right) = \lim_{x \to 0^+} x\left(\frac{4}{x} + Cx^{3/4}\right) = \lim_{x \to 0^+} (4 + Cx^{7/4}) = 4$$

Question 12

If $\frac{dy}{dx} + \frac{3}{\cos^2 x} y = \frac{1}{\cos^2 x}$, $x \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$ and $y\left(\frac{\pi}{4}\right) = \frac{4}{3}$, then $y\left(-\frac{\pi}{4}\right)$ equals

(2019 Main, 10 Jan I)

(a) $\frac{1}{3} + e^6$ (b) $-\frac{4}{3}$ (c) $\frac{1}{3} + e^3$ (d) $\frac{1}{3}$

Answer: (a)

Solution:

Given, differential equation is

$\frac{dy}{dx} + \left(\frac{3}{\cos^2 x}\right) y = \frac{1}{\cos^2 x}$, which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{3}{\cos^2 x}$ and $Q = \frac{1}{\cos^2 x}$.

Now, Integrating factor

$\text{IF} = e^{\int \frac{3}{\cos^2 x} dx} = e^{\int 3\sec^2 x\, dx} = e^{3\tan x}$ and the solution of differential equation is given by

$$y(\text{IF}) = \int (Q.(\text{IF}))\, dx$$

$$\Rightarrow \quad e^{3\tan x} . y = \int e^{3\tan x} \sec^2 x\, dx \quad \ldots(i)$$

Let $I = \int e^{3\tan x} \sec^2 x\, dx$

Put $3\tan x = t$

$\Rightarrow \quad 3\sec^2 x\, dx = dt$

$$\therefore \quad I = \int \frac{e^t}{3} dt = \frac{e^t}{3} + C = \frac{e^{3\tan x}}{3} + C$$

From Eq. (i)

$$e^{3\tan x} . y = \frac{e^{3\tan x}}{3} + C$$

It is given that when,

$$x = \frac{\pi}{4}, y \text{ is } \frac{4}{3}$$

$$\Rightarrow \quad e^3 \frac{4}{3} = \frac{e^3}{3} + C$$

$$\Rightarrow \quad C = e^3$$

Thus, $e^{3\tan x} y = \frac{e^{3\tan x}}{3} + e^3$

Now, when $x = -\frac{\pi}{4}$, $e^{-3} y = \frac{e^{-3}}{3} + e^3$

$$\Rightarrow \quad y = e^6 + \frac{1}{3} \quad \left[\because \tan\left(-\frac{\pi}{4}\right) = -1\right]$$

Question 13

If $y = y(x)$ is the solution of the differential equation, $x\frac{dy}{dx} + 2y = x^2$ satisfying $y(1) = 1$, then $y\left(\frac{1}{2}\right)$ is equal to

(2019 Main, 9 Jan I)

(a) $\frac{13}{16}$ (b) $\frac{1}{4}$ (c) $\frac{49}{16}$ (d) $\frac{7}{64}$

Answer: (c)

Solution:

Given differential equation can be rewritten as $\frac{dy}{dx} + \left(\frac{2}{x}\right) \cdot y = x$, which is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, where $P=\frac{2}{x}$ and $Q=x$

Now, integrating factor

$$(\text{IF})=e^{\int\frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$$

$[\because e^{\log f(x)}=f(x)]$

and the solution is given by

$$y(\text{IF})=\int (Q\times \text{IF})\,dx+C$$

$$\Rightarrow \quad yx^2=\int x^3\,dx+C$$

$$\Rightarrow \quad yx^2=\frac{x^4}{4}+C \quad \ldots\text{(i)}$$

Since, it is given that $y=1$ when $x=1$

$\therefore$ From Eq. (i), we get

$$1=\frac{1}{4}+C\Rightarrow C=\frac{3}{4} \quad \ldots\text{(ii)}$$

$$\therefore \quad 4x^2y=x^4+3 \quad \text{[using Eqs. (i) and (ii)]}$$

$$\Rightarrow \quad y=\frac{x^4+3}{4x^2}$$

$$\text{Now,} \quad y\left(\frac{1}{2}\right)=\frac{\frac{1}{16}+3}{4\times\frac{1}{4}}=\frac{49}{16}$$

Question 14

Let $y=y(x)$ be the solution of the differential equation $\sin x\frac{dy}{dx}+y\cos x=4x, x\in(0,\pi)$.

If $y\left(\frac{\pi}{2}\right)=0$, then $y\left(\frac{\pi}{6}\right)$ is equal to

(2018 Main)

(a) $\frac{4}{9\sqrt{3}}\pi^2$ (b) $\frac{-8}{9\sqrt{3}}\pi^2$ (c) $-\frac{8}{9}\pi^2$ (d) $-\frac{4}{9}\pi^2$

Answer: (c)

Solution:

We have,

$$\sin x\frac{dy}{dx}+y\cos x=4x\Rightarrow\frac{dy}{dx}+y\cot x=4x\operatorname{cosec}x$$

This is a linear differential equation of form

$$\frac{dy}{dx}+Py=Q$$

where $P=\cot x$, $Q=4x\operatorname{cosec}x$

Now, $IF=e^{\int Pdx}=e^{\int\cot x dx}=e^{\log\sin x}=\sin x$

Solution of the differential equation is

$$y\cdot\sin x=\int 4x\operatorname{cosec}x\sin x dx+C$$

$$\Rightarrow \quad y\sin x=\int 4xdx+C=2x^2+C$$

Put $x=\frac{\pi}{2}$, $y=0$, we get

$$C=-\frac{\pi^2}{2}\Rightarrow y\sin x=2x^2-\frac{\pi^2}{2}$$

$$\text{Put} \quad x=\frac{\pi}{6}$$

$$\therefore \quad y\left(\frac{1}{2}\right)=2\left(\frac{\pi^2}{36}\right)-\frac{\pi^2}{2}$$

$$\Rightarrow \quad y=\frac{\pi^2}{9}-\pi^2\Rightarrow y=-\frac{8\pi^2}{9}$$

Question 15

If a curve $y=f(x)$ passes through the point $(1,-1)$ and satisfies the differential equation, $y(1+xy)dx=x\,dy$, then $f\left(-\frac{1}{2}\right)$ is equal to

(2016 Main)

(a) $-\frac{2}{5}$ (b) $-\frac{4}{5}$ (c) $\frac{2}{5}$ (d) $\frac{4}{5}$

Answer: (d)

Solution:

Given differential equation is

$$y(1+xy)\,dx=x\,dy$$

$$\Rightarrow \quad y\,dx+xy^2\,dx=x\,dy$$

$$\Rightarrow \quad \frac{x\,dy-y\,dx}{y^2}=x\,dx$$

$$\Rightarrow \quad -\frac{(y\,dx-x\,dy)}{y^2}=x\,dx\Rightarrow -d\left(\frac{x}{y}\right)=x\,dx$$

On integrating both sides, we get

$$-\frac{x}{y}=\frac{x^2}{2}+C \quad \ldots\text{(i)}$$

$\because$ It passes through $(1,-1)$.

$$\therefore \quad 1=\frac{1}{2}+C\Rightarrow C=\frac{1}{2}$$

Now, from Eq. (i) $-\frac{x}{y}=\frac{x^2}{2}+\frac{1}{2}$

$$\Rightarrow \quad x^2+1=-\frac{2x}{y}$$

$$\Rightarrow \quad y=-\frac{2x}{x^2+1}$$

$$\therefore \quad f\left(-\frac{1}{2}\right)=\frac{4}{5}$$

Question 16

Let $y(x)$ be the solution of the differential equation $(x\log x)\frac{dy}{dx}+y=2x\log x, (x\geq 1)$. Then, $y(e)$ is equal to

(2015 Main)

(a) e (b) 0 (c) 2 (d) $2e$

Answer: (c)

[P.T.O. for Sol'n]

Solution:

Given differential equation is

$$(x\log x)\frac{dy}{dx}+y=2x\log x$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y}{x\log x}=2$$

This is a linear differential equation.

$$\therefore \quad \text{IF}=e^{\int\frac{1}{x\log x}dx}=e^{\log(\log x)}=\log x$$

Now, the solution of given differential equation is given by

$$y\cdot\log x=\int\log x\cdot 2dx$$

$$\Rightarrow \quad y\cdot\log x=2\int\log x\,dx$$

$$\Rightarrow \quad y\cdot\log x=2\ [x\log x-x]+c$$

At $x=1 \Rightarrow c=2$

$$\Rightarrow \quad y\cdot\log x=2\ [x\log x-x]+2$$

At $x=e,\ y=2(e-e)+2$

$$\Rightarrow \quad y=2$$

Question 17

The function $y=f(x)$ is the solution of the differential equation $\frac{dy}{dx}+\frac{xy}{x^2-1}=\frac{x^4+2x}{\sqrt{1-x^2}}$ in $(-1,1)$ satisfying $f(0)=0$. Then, $\int_{-\frac{\sqrt{3}}{2}}^{\sqrt{3}/2}f(x)\,dx$ is **(2014 Adv.)**

(a) $\frac{\pi}{3}-\frac{\sqrt{3}}{2}$ (b) $\frac{\pi}{3}-\frac{\sqrt{3}}{4}$ (c) $\frac{\pi}{6}-\frac{\sqrt{3}}{4}$ (d) $\frac{\pi}{6}-\frac{\sqrt{3}}{2}$

Answer: (b)

Solution:

PLAN (i) Solution of the differential equation $\frac{dy}{dx}+Py=Q$ is

$$y\cdot(\text{IF})=\int Q\cdot(\text{IF})\,dx+c$$

where, $\text{IF}=e^{\int P dx}$

(ii) $\int_{-a}^{a}f(x)\,dx=2\int_0^a f(x)\,dx$, if $f(-x)=f(x)$

Given differential equation

$$\frac{dy}{dx}+\frac{x}{x^2-1}y=\frac{x^4+2x}{\sqrt{1-x^2}}$$

This is a linear differential equation.

$$\text{IF}=e^{\int\frac{x}{x^2-1}dx}=e^{\frac{1}{2}\ln|x^2-1|}=\sqrt{1-x^2}$$

$$\Rightarrow \text{Solution is } y\sqrt{1-x^2}=\int\frac{x(x^3+2)}{\sqrt{1-x^2}}\cdot\sqrt{1-x^2}\,dx$$

or $\quad y\sqrt{1-x^2}=\int(x^4+2x)\,dx=\frac{x^5}{5}+x^2+c$

$$f(0)=0 \Rightarrow c=0 \Rightarrow f(x)\sqrt{1-x^2}=\frac{x^5}{5}+x^2$$

Now, $\int_{-\sqrt{3}/2}^{\sqrt{3}/2}f(x)\,dx=\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\frac{x^2}{\sqrt{1-x^2}}\,dx$ [using property]

$$=2\int_0^{\sqrt{3}/2}\frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$=2\int_0^{\pi/3}\frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \qquad [\text{taking } x=\sin\theta]$$

$$=2\int_0^{\pi/3}\sin^2\theta\,d\theta=\int_0^{\pi/3}(1-\cos2\theta)\,d\theta$$

$$=\left(\theta-\frac{\sin2\theta}{2}\right)_0^{\pi/3}=\frac{\pi}{3}-\frac{\sin2\pi/3}{2}=\frac{\pi}{3}-\frac{\sqrt{3}}{4}$$

Question 18

Let $f:[1/2,1]\to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x)<2f(x)$ and $f(1/2)=1$. Then, the value of $\int_{1/2}^{1}f(x)\,dx$ lies in the interval **(2013 Adv.)**

(a) $(2e-1,2e)$ (b) $(e-1,2e-1)$

(c) $\left(\frac{e-1}{2},e-1\right)$ (d) $\left(0,\frac{e-1}{2}\right)$

Answer: (d)

Solution:

PLAN Whenever we have linear differential equation containing inequality, we should always check for increasing or decreasing,

i.e. for $\frac{dy}{dx}+Py<0 \Rightarrow \frac{dy}{dx}+Py>0$

Multiply by integrating factor, i.e. $e^{\int Pdx}$ and convert into total differential equation.

Here, $f'(x)<2f(x)$, multiplying by $e^{-\int 2dx}$

$$f'(x)\cdot e^{-2x}-2e^{-2x}f(x)<0 \Rightarrow \frac{d}{dx}(f(x)\cdot e^{-2x})<0$$

$\therefore\ \phi(x)=f(x)e^{-2x}$ is decreasing for $x\in\left[\frac{1}{2},1\right]$

Thus, when $x>\frac{1}{2}$

$$\phi(x)<\phi\left(\frac{1}{2}\right) \Rightarrow e^{-2x}f(x)<e^{-1}\cdot f\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad f(x)<e^{2x-1}\cdot1,\quad \text{given } f\left(\frac{1}{2}\right)=1$$

$$\Rightarrow \quad 0<\int_{1/2}^{1}f(x)\,dx<\int_{1/2}^{1}e^{2x-1}\,dx$$

$$\Rightarrow \quad 0<\int_{1/2}^{1}f(x)\,dx<\left(\frac{e^{2x-1}}{2}\right)_{1/2}^{1}$$

$$\Rightarrow \quad 0<\int_{1/2}^{1}f(x)\,dx<\frac{e-1}{2}$$

Question 19

Let $f(x)$ be differentiable on the interval $(0, \infty)$ such that $f(1)=1$, and $\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$. Then, $f(x)$ is **(2007, 3M)**

(a) $\frac{1}{3x} + \frac{2x^2}{3}$ (b) $-\frac{1}{3x} + \frac{4x^2}{3}$

(c) $-\frac{1}{x} + \frac{2}{x^2}$ (d) $\frac{1}{x}$

Answer: (a)

Solution:

Given, $\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$

$\Rightarrow \quad x^2 f'(x) - 2x f(x) + 1 = 0$

$\Rightarrow \quad \frac{x^2 f'(x) - 2x f(x)}{(x^2)^2} + \frac{1}{x^4} = 0$

$\Rightarrow \quad \frac{d}{dx}\left(\frac{f(x)}{x^2}\right) = -\frac{1}{x^4}$

On integrating both sides, we get

$$f(x) = cx^2 + \frac{1}{3x}$$

Also, $f(1) = 1$, $\quad c = \frac{2}{3}$

Hence, $\quad f(x) = \frac{2}{3}x^2 + \frac{1}{3x}$

Question 20

If $x\,dy = y\,(dx + y\,dy)$, $y(1) = 1$ and $y(x) > 0$. Then, $y(-3)$ is equal to **(2005, 1M)**

(a) 3 (b) 2

(c) 1 (d) 0

Answer: (a)

Solution:

Given, $x\,dy = y(dx + y\,dy), y > 0$

$\Rightarrow \quad x\,dy - y\,dx = y^2 dy$

$\Rightarrow \quad \frac{x\,dy - y\,dx}{y^2} = dy \Rightarrow d\left(\frac{x}{y}\right) = -dy$

On integrating both sides, we get

$$\frac{x}{y} = -y + c \qquad \dots(i)$$

Since, $\quad y(1) = 1 \quad \Rightarrow x = 1, y = 1$

$\therefore \quad c = 2$

Now, Eq. (i) becomes, $\frac{x}{y} + y = 2$

Again, for $x = -3$

$\Rightarrow \quad -3 + y^2 = 2y$

$\Rightarrow \quad y^2 - 2y - 3 = 0$

$\Rightarrow \quad (y+1)(y-3) = 0$

As $y > 0$, take $y = 3$, neglecting $y = -1$.

Question 21

If $y(t)$ is a solution of $(1+t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$, then $y(1)$ is equal to **(2003, 1M)**

(a) $-1/2$ (b) $e + 1/2$

(c) $e - 1/2$ (d) $1/2$

Answer: (a)

Solution:

Given, $\frac{dy}{dt} - \left(\frac{t}{1+t}\right) y = \frac{1}{(1+t)}$ and $y(0) = -1$

Which represents linear differential equation of first order.

$\therefore \quad \text{IF} = e^{\int -\left(\frac{t}{1+t}\right) dt} = e^{-t + \log(1+t)} = e^{-t} \cdot (1+t)$

Required solution is,

$ye^{-t}(1+t) = \int \frac{1}{1+t} \cdot e^{-t}(1+t)\,dt + c = \int e^{-t}\,dt + c$

$\Rightarrow \quad ye^{-t}(1+t) = -e^{-t} + c$

Since, $\quad y(0) = -1$

$\Rightarrow \quad -1 \cdot e^0 (1+0) = -e^0 + c$

$c = 0$

$\therefore \quad y = -\frac{1}{(1+t)} \Rightarrow y(1) = -\frac{1}{2}$

Question 22

Let $f : (0, \infty) \to R$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then **(2016 Adv.)**

(a) $\lim_{x \to 0+} f'\left(\frac{1}{x}\right) = 1$

(b) $\lim_{x \to 0+} x f\left(\frac{1}{x}\right) = 2$

(c) $\lim_{x \to 0+} x^2 f'(x) = 0$

(d) $|f(x)| \le 2$ for all $x \in (0, 2)$

Answer: (a)

Solution:

Here, $f'(x) = 2 - \frac{f(x)}{x}$

or $\quad \frac{dy}{dx} + \frac{y}{x} = 2$ [i.e. linear differential equation in y]

Integrating Factor, IF $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$\therefore$ Required solution is $y \cdot (\text{IF}) = \int Q(\text{IF}) dx + C$

$\Rightarrow \quad y(x) = \int 2(x)\, dx + C$

$\Rightarrow \quad yx = x^2 + C$

$\therefore \quad y = x + \dfrac{C}{x} \qquad [\because C \neq 0, \text{ as } f(1) \neq 1]$

(a) $\lim\limits_{x \to 0^+} f'\left(\dfrac{1}{x}\right) = \lim\limits_{x \to 0^+} (1 - Cx^2) = 1$

$\therefore$ Option (a) is correct.

(b) $\lim\limits_{x \to 0^+} x f\left(\dfrac{1}{x}\right) = \lim\limits_{x \to 0^+} (1 + Cx^2) = 1$

$\therefore$ Option (b) is incorrect.

(c) $\lim\limits_{x \to 0^+} x^2 f'(x) = \lim\limits_{x \to 0^+} (x^2 - C) = -C \neq 0$

$\therefore$ Option (c) is incorrect.

(d) $f(x) = x + \dfrac{C}{x}, C \neq 0$

For $C > 0$, $\lim\limits_{x \to 0^+} f(x) = \infty$

$\therefore$ Function is not bounded in (0, 2).

$\therefore$ Option (d) is incorrect.

Question 23

If $y(x)$ satisfies the differential equation $y' - y \tan x = 2x \sec x$ and $y(0)$, then **(2012)**

(a) $y\left(\dfrac{\pi}{4}\right) = \dfrac{\pi^2}{8\sqrt{2}}$ (b) $y'\left(\dfrac{\pi}{4}\right) = \dfrac{\pi^2}{18}$

(c) $y\left(\dfrac{\pi}{3}\right) = \dfrac{\pi^2}{9}$ (d) $y'\left(\dfrac{\pi}{3}\right) = \dfrac{4\pi}{3} + \dfrac{2\pi^2}{3\sqrt{3}}$

Answer: (a, d)

Solution:

PLAN Linear differential equation under one variable.

$$\frac{dy}{dx} + Py = Q; \quad \text{IF} = e^{\int P dx}$$

$\therefore$ Solution is, $y(\text{IF}) = \int Q \cdot (\text{IF}) dx + C$

$$y' - y \tan x = 2x \sec x \text{ and } y(0) = 0$$

$\Rightarrow \quad \dfrac{dy}{dx} - y \tan x = 2x \sec x$

$\therefore \quad \text{IF} = \int e^{-\tan x} dx = e^{\log|\cos x|} = \cos x$

Solution is $y \cdot \cos x = \int 2x \sec x \cdot \cos x\, dx + C$

$\Rightarrow \quad y \cdot \cos x = x^2 + C$

As $\quad y(0) = 0 \Rightarrow C = 0$

$\therefore \quad y = x^2 \sec x$

Now, $\quad y\left(\dfrac{\pi}{4}\right) = \dfrac{\pi^2}{8\sqrt{2}}$

$\Rightarrow \quad y'\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{\sqrt{2}} + \dfrac{\pi^2}{8\sqrt{2}}$

$$y\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{9} \Rightarrow y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$$

Question 24

Let $u(x)$ and $v(x)$ satisfy the differential equations $\dfrac{du}{dx} + p(x)u = f(x)$ and $\dfrac{dv}{dx} + p(x)v = g(x)$, where $p(x), f(x)$ and $g(x)$ are continuous functions. If $u(x_1) > v(x_1)$ for some x_1 and $f(x) > g(x)$ for all $x > x_1$, prove that any point (x, y) where $x > x_1$ does not satisfy the equations $y = u(x)$ and $y = v(x)$. **(1997, 5M)**

Solution:

Let $\quad w(x) = u(x) - v(x) \qquad$ ...(i)

and $\quad h(x) = f(x) - g(x)$

On differentiating Eq. (i) w.r.t. x

$\dfrac{dw}{dx} = \dfrac{du}{dx} - \dfrac{dv}{dx}$

$= \{f(x) - p(x) \cdot u(x)\} - \{g(x) - p(x)\, v(x)\} \qquad$ [given]

$= \{f(x) - g(x)\} - p(x)[u(x) - v(x)]$

$\Rightarrow \dfrac{dw}{dx} = h(x) - p(x) \cdot w(x) \qquad$ ...(ii)

$\Rightarrow \dfrac{dw}{dx} + p(x)\, w(x) = h(x)$ which is linear differential equation .

The integrating factor is given by

$$\text{IF} = e^{\int p(x) dx} = r(x) \qquad \text{[let]}$$

On multiplying both sides of Eq. (ii) of $r(x)$, we get

$r(x) \cdot \dfrac{dw}{dx} + p(x)(r(x))\, w(x) = r(x) \cdot h(x)$

$\Rightarrow \dfrac{d}{dx}[r(x)\, w(x)] = r(x) \cdot h(x) \qquad \left[\because \dfrac{dr}{dx} = p(x) \cdot r(x)\right]$

Now, $\quad r(x) = e^{\int P(x) dx} > 0, \forall x$

and $\quad h(x) = f(x) - g(x) > 0$, for $x > x_1$

Thus, $\quad \dfrac{d}{dx}[r(x)\, w(x)] > 0, \ \forall x > x_1$

$r(x)\, w(x)$ increases on the interval $[x, \infty[$

Therefore, for all $x > x_1$

$r(x)\, w(x) > r(x_1)\, w(x_1) > 0$

$[\because r(x_1) > 0 \text{ and } u(x_1) > v(x_1)]$

$\Rightarrow \quad w(x) > 0 \ \forall x > x_1$

$\Rightarrow \quad u(x) > v(x) \ \forall x > x_1 \qquad [\because r(x) > 0]$

Hence, there cannot exist a point (x, y) such that $x > x_1$ and $y = u(x)$ and $y = v(x)$.

Question 25

Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\dfrac{d\,f(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then, the value of $y(2)$ is ...... **(2011)**

Answer: 0

Solution:

$$\frac{dy}{dx} + y \cdot g'(x) = g(x)\, g'(x)$$

$$\text{IF} = e^{\int g'(x)\,dx} = e^{g(x)}$$

$\therefore$ Solution is $y\,(e^{g(x)}) = \int g(x) \cdot g'(x) \cdot e^{g(x)}\, dx + C$

Put $\quad g(x) = t,\ g'(x)\,dx = dt$

$$y(e^{g(x)}) = \int t \cdot e^t\, dt + C$$

$$= t \cdot e^t - \int 1 \cdot e^t\, dt + C = t \cdot e^t - e^t + C$$

$$y\,e^{g(x)} = (g(x) - 1)\, e^{g(x)} + C \qquad \text{...(i)}$$

Given, $\quad y(0) = 0,\ g(0) = g(2) = 0$

$\therefore$ Eq. (i) becomes,

$$y(0) \cdot e^{g(0)} = (g(0) - 1) \cdot e^{g(0)} + C$$

$\Rightarrow \quad 0 = (-1) \cdot 1 + C \Rightarrow C = 1$

$\therefore \quad y(x) \cdot e^{g(x)} = (g(x) - 1)\, e^{g(x)} + 1$

$\Rightarrow \quad y(2) \cdot e^{g(2)} = (g(2) - 1)\, e^{g(2)} + 1$, where $g(2) = 0$

$\Rightarrow \quad y(2) \cdot 1 = (-1) \cdot 1 + 1$

$$y(2) = 0$$