Question 1

Given that the slope of the tangent to a curve $y = y(x)$ at any point (x, y) is $\frac{2y}{x^2}$. If the curve passes through the centre of the circle $x^2 + y^2 - 2x - 2y = 0$, then its equation is (2019 Main, 8 April II)

(a) $x^2 \log_e|y| = -2(x-1)$ (b) $x \log_e|y| = x - 1$
(c) $x \log_e|y| = 2(x-1)$ (d) $x \log_e|y| = -2(x-1)$

Answer: (c)

Given, $\frac{dy}{dx} = \frac{2y}{x^2}$

$\Rightarrow \int \frac{dy}{y} = \int \frac{2}{x^2} dx$ [integrating both sides]

$\Rightarrow \log_e|y| = -\frac{2}{x} + C$ ...(i)

Since, curve (i) passes through centre (1, 1) of the circle

$x^2 + y^2 - 2x - 2y = 0$

$\therefore \log_e(1) = -\frac{2}{1} + C \Rightarrow C = 2$

$\therefore$ Equation required curve is

$\log_e|y| = -\frac{2}{x} + 2$ [put $C = 2$ in Eq. (i)]

$\Rightarrow x \log_e|y| = 2(x-1)$

Question 2

The curve amongst the family of curves represented by the differential equation, $(x^2 - y^2)dx + 2xy\,dy = 0$, which passes through (1, 1), is (2019 Main, 10 Jan II)

(a) a circle with centre on the Y-axis
(b) a circle with centre on the X-axis
(c) an ellipse with major axis along the Y-axis
(d) a hyperbola with transverse axis along the X-axis.

Answer: (b)

Given differential equation is

$(x^2 - y^2)dx + 2xy\,dy = 0$, which can be written as

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Put $y = vx$ [$\because$ it is in homogeneous form]

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, differential equation becomes

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x(vx)} \Rightarrow v + x\frac{dv}{dx} = \frac{(v^2-1)x^2}{2vx^2}$$

$\Rightarrow x\frac{dv}{dx} = \frac{v^2-1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v}$

$\Rightarrow x\frac{dv}{dx} = -\frac{1+v^2}{2v} \Rightarrow \int \frac{2v\,dv}{1+v^2} = -\int \frac{dx}{x}$

$\Rightarrow \ln(1+v^2) = -\ln x - \ln C$

$\left[\because \int \frac{f'(x)}{f(x)}dx \Rightarrow \ln|f(x)| + C\right]$

$\Rightarrow \ln|(1+v^2)Cx| = 0$ [$\because \ln A + \ln B = \ln AB$]

$\Rightarrow (1+v^2)Cx = 1$ [$\log_e x = 0 \Rightarrow x = e^0 = 1$]

Now, putting $v = \frac{y}{x}$, we get

$$\left(1 + \frac{y^2}{x^2}\right)Cx = 1 \Rightarrow C(x^2 + y^2) = x$$

$\because$ The curve passes through (1, 1), so

$$C(1+1) = 1 \Rightarrow C = \frac{1}{2}$$

Thus, required curve is $x^2 + y^2 - 2x = 0$, which represent a circle having centre (1, 0)

$\therefore$ The solution of given differential equation represents a circle with centre on the X-axis.

Question 3

Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. If $p(0) = 100$, then $p(t)$ is equal to (2014 Main)

(a) $400 - 300e^{\frac{t}{2}}$ (b) $300 - 200e^{-\frac{t}{2}}$
(c) $600 - 500e^{\frac{t}{2}}$ (d) $400 - 300e^{-\frac{t}{2}}$

Answer: (a)

Given, differential equation is $\frac{dp}{dt} - \frac{1}{2}p(t) = -200$ is a linear differential equation.

Here, $p(t) = \frac{-1}{2}, Q(t) = -200$

$$\text{IF} = e^{\int -\left(\frac{1}{2}\right)dt} = e^{-\frac{t}{2}}$$

Hence, solution is

$$p(t) \cdot \text{IF} = \int Q(t) \cdot \text{IF}\,dt$$

$$p(t) \cdot e^{-\frac{t}{2}} = \int -200 \cdot e^{-\frac{t}{2}}dt$$

$$p(t) \cdot e^{-\frac{t}{2}} = 400\,e^{-\frac{t}{2}} + K$$

$\Rightarrow p(t) = 400 + ke^{-1/2}$

If $p(0) = 100$, then $k = -300$

$\Rightarrow p(t) = 400 - 300\,e^{\frac{t}{2}}$

Question 4

A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right), x > 0.$

Then, the equation of the curve is **(2013 Adv.)**

(a) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (b) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(c) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (d) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

Answer: (a)

PLAN To solve homogeneous differential equation, i.e. substitute $\frac{y}{x} = v$

$\therefore \quad y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Here, slope of the curve at (x, y) is

$$\frac{dy}{dx} = \frac{y}{x} + \sec\left(\frac{y}{x}\right)$$

Put $\quad \frac{y}{x} = v$

$$\therefore \quad v + x\frac{dv}{dx} = v + \sec(v) \Rightarrow x\frac{dv}{dx} = \sec(v)$$

$$\Rightarrow \quad \int \frac{dv}{\sec v} = \int \frac{dx}{x} \Rightarrow \int \cos v\, dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \sin v = \log x + \log c \Rightarrow \sin\left(\frac{y}{x}\right) = \log(cx)$$

As it passes through $\left(1, \frac{\pi}{6}\right) \Rightarrow \sin\left(\frac{\pi}{6}\right) = \log c$

$$\Rightarrow \quad \log c = \frac{1}{2}$$

$$\therefore \quad \sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$$

Question 5

At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P with respect to additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employees 25 more workers, then the new level of production of items is **(2013 Main)**

(a) 2500 (b) 3000 (c) 3500 (d) 4500

Answer: (c)

Given, $\frac{dP}{dx} = (100 - 12\sqrt{x}) \Rightarrow dP = (100 - 12\sqrt{x})\, dx$

On integrating both sides, we get

$$\int dP = \int (100 - 12\sqrt{x})\, dx$$

$$P = 100x - 8x^{3/2} + C$$

When $x = 0$, then $P = 2000 \Rightarrow C = 2000$

Now, when $x = 25$, then is

$$P = 100 \times 25 - 8 \times (25)^{3/2} + 2000$$
$$= 2500 - 8 \times 125 + 2000$$
$$= 4500 - 1000 = 3500$$

Question 6

A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0, x > 0$, passes through the point $(1, 3)$. Then, the solution curve **(2016 Adv.)**

(a) intersects $y = x + 2$ exactly at one point
(b) intersects $y = x + 2$ exactly at two points
(c) intersects $y = (x + 2)^2$
(d) does not intersect $y = (x + 3)^2$

Answer: (a, d)

Given, $\quad (x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0$

$$\Rightarrow \quad [(x^2 + 4x + 4) + y(x + 2)]\frac{dy}{dx} - y^2 = 0$$

$$\Rightarrow \quad [(x+2)^2 + y(x+2)]\frac{dy}{dx} - y^2 = 0$$

Put $x + 2 = X$ and $y = Y$, then

$$(X^2 + XY)\frac{dY}{dX} - Y^2 = 0$$

$$\Rightarrow \quad X^2 dY + XY dY - Y^2 dX = 0$$

$$\Rightarrow \quad X^2 dY + Y(X dY - Y dX) = 0$$

$$\Rightarrow \quad -\frac{dY}{Y} = \frac{X dY - Y dX}{X^2}$$

$$\Rightarrow \quad -d(\log|Y|) = d\left(\frac{Y}{X}\right)$$

On integrating both sides, we get

$$-\log|Y| = \frac{Y}{X} + C, \text{ where } x + 2 = X \text{ and } y = Y$$

$$\Rightarrow \quad -\log|y| = \frac{y}{x+2} + C \quad \ldots\text{(i)}$$

Since, it passes through the point $(1, 3)$.

$$\therefore \quad -\log 3 = 1 + C$$

$$\Rightarrow \quad C = -1 - \log 3 = -(\log e + \log 3)$$
$$= -\log 3e$$

$\therefore$ Eq. (i) becomes

$$\log|y| + \frac{y}{x+2} - \log(3e) = 0$$

$$\Rightarrow \quad \log\left(\frac{|y|}{3e}\right) + \frac{y}{x+2} = 0 \quad \ldots\text{(ii)}$$

Now, to check option (a), $y = x + 2$ intersects the curve.

$$\Rightarrow \quad \log\left(\frac{|x+2|}{3e}\right) + \frac{x+2}{x+2} = 0 \Rightarrow \log\left(\frac{|x+2|}{3e}\right) = -1$$

$$\Rightarrow \quad \frac{|x+2|}{3e} = e^{-1} = \frac{1}{e}$$

$\Rightarrow \quad |x+2| = 3$ or $x+2 = \pm 3$

$\therefore x = 1, -5$ (rejected), as $x > 0$ [given]

$\therefore x = 1$ only one solution.

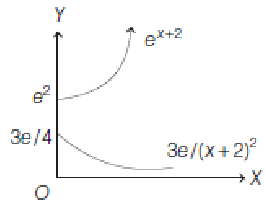
Thus, (a) is the correct answer.

To check option (c), we have

$y = (x+2)^2$ and $\log\left(\frac{|y|}{3e}\right) + \frac{y}{x+2} = 0$

$$\Rightarrow \log\left[\frac{|x+2|^2}{3e}\right] + \frac{(x+2)^2}{x+2} = 0 \Rightarrow \log\left[\frac{|x+2|^2}{3e}\right] = -(x+2)$$

$$\Rightarrow \frac{(x+2)^2}{3e} = e^{-(x+2)} \text{ or } (x+2)^2 \cdot e^{x+2} = 3e \Rightarrow e^{x+2} = \frac{3e}{(x+2)^2}$$



Clearly, they have no solution.

To check option (d), $y = (x+3)^2$

i.e. $\log\left[\frac{|x+3|^2}{3e}\right] + \frac{(x+3)^2}{(x+2)} = 0$

To check the number of solutions.

Let $g(x) = 2\log(x+3) + \frac{(x+3)^2}{(x+2)} - \log(3e)$

$$\therefore \quad g'(x) = \frac{2}{x+3} + \left(\frac{(x+2)\cdot 2(x+3) - (x+3)^2 \cdot 1}{(x+2)^2}\right) - 0$$

$$= \frac{2}{x+3} + \frac{(x+3)(x+1)}{(x+2)^2}$$

Clearly, when $x > 0$, then, $g'(x) > 0$

$\therefore \quad g(x)$ is increasing, when $x > 0$.

Thus, when $x > 0$, then $g(x) > g(0)$

$$g(x) > \log\left(\frac{3}{e}\right) + \frac{9}{4} > 0$$

Hence, there is no solution. Thus, option (d) is true.

Question 7

Tangent is drawn at any point P of a curve which passes through (1, 1) cutting X-axis and Y-axis at A and B, respectively. If $BP : AP = 3:1$, then **(2006, 3M)**

(a) differential equation of the curve is $3x\frac{dy}{dx} + y = 0$

(b) differential equation of the curve is $3x\frac{dy}{dx} - y = 0$

(c) curve is passing through $\left(\frac{1}{8}, 2\right)$

(d) normal at (1, 1) is $x + 3y = 4$.

Answer: (a, c)

Since, $BP : AP = 3:1$. Then, equation of tangent is

$$Y - y = f'(x)(X - x)$$

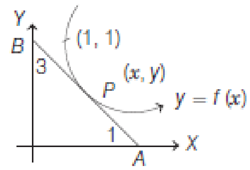
The intercept on the coordinate axes are

$$A\left(x - \frac{y}{f'(x)}, 0\right)$$

and $\quad B[0, y - x f'(x)]$

Since, P is internally intercepts a line AB,

$$\therefore \quad x = \frac{3\left(x - \frac{y}{f'(x)}\right) + 1 \times 0}{3+1}$$



$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{-3x} \quad \Rightarrow \frac{dy}{y} = -\frac{1}{3x}dx$$

On integrating both sides, we get

$$xy^3 = c$$

Since, curve passes through (1, 1), then $c = 1$.

$\therefore \quad xy^3 = 1$

At $\quad x = \frac{1}{8} \Rightarrow \quad y = 2$

Hence, (a) and (c) are correct answers.

Question 8

A spherical rain drop evaporates at a rate proportional to its surface area at any instant t. The differential equation giving the rate of change of the rains of the rain drop is ..... . **(1997C, 2M)**

Solution:

Since, rate of change of volume $\propto$ surface area

$$\Rightarrow \quad \frac{dV}{dt} \propto \text{SA}$$

$$\Rightarrow \quad 4\pi r^2 \cdot \frac{dr}{dt} = -\lambda 4\pi r^2$$

$\frac{dr}{dt} = -\lambda$ is required differential equation.

Question 9

If length of tangent at any point on the curve $y = f(x)$ intercepted between the point and the X-axis is of length 1. Find the equation of the curve. **(2005, 4M)**

Solution:

Since, the length of tangent $= \left| y\sqrt{1+\left(\frac{dx}{dy}\right)^2} \right| = 1$

$$\Rightarrow \quad y^2\left(1+\left(\frac{dx}{dy}\right)^2\right) = 1$$

$$\therefore \quad \frac{dy}{dx} = \pm \frac{y}{\sqrt{1-y^2}}$$

$$\Rightarrow \quad \int \frac{\sqrt{1-y^2}}{y}\, dy = \pm \int x\, dx$$

$$\Rightarrow \quad \int \frac{\sqrt{1-y^2}}{y}\, dy = \pm x + C$$

Put $y = \sin\theta \Rightarrow dy = \cos\theta\, d\theta$

$$\therefore \quad \int \frac{\cos\theta}{\sin\theta}\cdot \cos\theta\, d\theta = \pm x + C$$

$$\Rightarrow \quad \int \frac{\cos^2\theta}{\sin^2\theta}\cdot \sin\theta\, d\theta = \pm x + C$$

Again put $\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt$

$$\therefore \quad -\int \frac{t^2}{1-t^2}\, dt = \pm x + C$$

$$\Rightarrow \quad \int \left(1 - \frac{1}{1-t^2}\right) dt = \pm x + C$$

$$\Rightarrow \quad t - \log\left|\frac{1+t}{1-t}\right| = \pm x + C$$

$$\Rightarrow \sqrt{1-y^2} - \log\left|\frac{1+\sqrt{1-y^2}}{1-\sqrt{1-y^2}}\right| = \pm x + C$$

Question 10

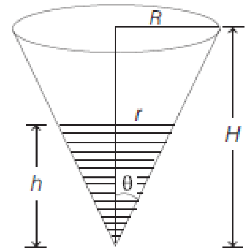
A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant $= k > 0$). Find the time after which the cone is empty. **(2003, 4M)**

Solution:

Given, liquid evaporates at a rate proportional to its surface area.

$$\Rightarrow \quad \frac{dV}{dt} \propto -S \qquad \ldots(i)$$

We know that, volume of cone $= \frac{1}{3}\pi r^2 h$



and surface area $= \pi r^2$

$$\text{or} \quad V = \frac{1}{3}\pi r^2 h \quad \text{and} \quad S = \pi r^2 \qquad \ldots(ii)$$

$$\text{Where,} \quad \tan\theta = \frac{R}{H} \quad \text{and} \quad \frac{r}{h} = \tan\theta \qquad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$V = \frac{1}{3}\pi r^3 \cot\theta \quad \text{and} \quad S = \pi r^2 \qquad \ldots(iv)$$

On substituting Eq. (iv) in Eq. (i), we get

$$\frac{1}{3}\cot\theta \cdot 3r^2 \frac{dr}{dt} = -k\pi r^2$$

$$\Rightarrow \quad \cot\theta \int_R^0 dr = -k\int_0^T dt$$

$$\Rightarrow \quad \cot\theta\, (0 - R) = -k\,(T - 0)$$

$$\Rightarrow \quad R\cot\theta = kT \Rightarrow H = kT \quad \text{[from Eq. (iii)]}$$

$$\Rightarrow \quad T = \frac{H}{k}$$

$\therefore$ Required time after which the cone is empty, $T = \frac{H}{k}$

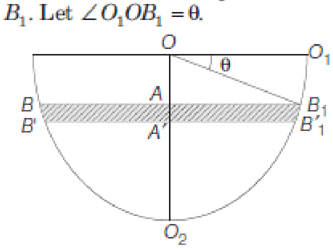
Question 11

A hemispherical tank of radius 2 m is initially full of water and has an outlet of 12 cm^2 cross-sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $v(t) = 0.6\sqrt{2gh(t)}$, where $v(t)$ and $h(t)$ are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time t and g is the acceleration due to gravity. Find the time it takes to empty the tank. **(2001, 10M)**

Hint Form a differential equation by relating the decreases of water level to the outflow.

Solution:

Let O be the centre of hemispherical tank. Let at any instant t, water level be BAB_1 and at $t + dt$, water level is $B'A'B_1'$. Let $\angle O_1OB_1 = \theta$.



$\Rightarrow AB_1 = r\cos\theta$ and $OA = r\sin\theta$ decrease in the water volume in time $dt = \pi\, AB_1^2 \cdot d\,(OA)$

[πr^2 is surface area of water level and $d\,(OA)$ is depth of water level]

$$= \pi r^2 \cdot \cos^2\theta \cdot r\cos\theta\, d\theta$$
$$= \pi r^3 \cdot \cos^3\theta\, d\theta$$

Also, $h\,(t) = O_2A = r - r\sin\theta = r\,(1-\sin\theta)$

Now, outflow rate $Q = A\cdot v\,(t) = A\cdot 0.6\sqrt{2gr\,(1-\sin\theta)}$

Where, A is the area of the outlet.

Thus, volume flowing out in time dt.

$$\Rightarrow \quad Q\,dt = A\cdot(0.6)\cdot\sqrt{2gr}\cdot\sqrt{1-\sin\theta}\,dt$$

We have, $\pi r^3\cos^3\theta\,d\theta = A\,(0.6)\cdot\sqrt{2gr}\cdot\sqrt{1-\sin\theta}\,dt$

$$\Rightarrow \frac{\pi r^3}{A\,(0.6)\sqrt{2gr}}\cdot\frac{\cos^3\theta}{\sqrt{(1-\sin\theta)}}\,d\theta = dt$$

Let the time taken to empty the tank be T.

$$\text{Then,}\quad T = \int_0^{\pi/2}\frac{\pi r^3}{A\,(0.6)\cdot\sqrt{2gr}}\cdot\frac{\cos^3\theta}{\sqrt{1-\sin\theta}}\,d\theta$$
$$= \frac{-\pi r^3}{A\,(0.6)\sqrt{2gr}}\int_0^{\pi/2}\frac{1-\sin^2\theta\,(-\cos\theta)}{\sqrt{1-\sin\theta}}\,d\theta$$

Let $\quad t_1 = \sqrt{1-\sin\theta}$

$$\Rightarrow \quad dt_1 = \frac{-\cos\theta}{\sqrt{1-\sin\theta}}\,d\theta$$

$$\therefore \quad T = \frac{-2\pi r^3}{A\,(0.6)\sqrt{2gr}}\int_1^0[1-(1-t_1^2)^2]\,dt_1$$

$$\Rightarrow \quad T = \frac{-2\pi r^3}{A\,(0.6)\sqrt{2gr}}\int_1^0[1-(1+t_1^4-2t_1^2)]\,dt_1$$

$$\Rightarrow \quad T = \frac{-2\pi r^3}{A\,(0.6)\sqrt{2gr}}\int_1^0[1-1-t_1^4+2t_1^2]\,dt_1$$

$$\Rightarrow \quad T = \frac{2\pi r^3}{A\,(0.6)\sqrt{2gr}}\int_1^0(t_1^4-2t_1^2)\,dt_1$$

$$\Rightarrow \quad T = \frac{2\pi r^3}{A\,(0.6)\sqrt{2gr}}\left[\frac{t_1^5}{5}-\frac{2t_1^3}{3}\right]_1^0$$

$$\Rightarrow \quad T = \frac{2\pi\cdot r^{5/2}}{A\left(\frac{6}{10}\right)\sqrt{2g}}\cdot\left[0-\frac{1}{5}-0+\frac{2}{3}\right]$$

$$\Rightarrow \quad T = \frac{2\pi\cdot 2^{5/2}\,(10^2)^{5/2}}{12\cdot\frac{3}{5}\cdot\sqrt{2}\cdot\sqrt{g}}\left[\frac{2}{3}-\frac{1}{5}\right]$$
$$= \frac{2\pi\times 10^5\cdot 4\cdot 5}{(12\times 3)\sqrt{g}}\left[\frac{10-3}{15}\right]$$
$$= \frac{2\pi\times 10^5\times 7}{3\cdot 3\cdot\sqrt{g}\cdot 3} = \frac{14\pi\times 10^5}{27\sqrt{g}}\text{ unit}$$

Question 12

A country has food deficit of 10%. Its population grows continuously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self- sufficient in food after n years, where n is the smallest integer bigger than or equal to $\dfrac{\ln 10 - \ln 9}{\ln\,(1.04)-(0.03)}$.

(2000, 10M)

Solution:

Let X_0 be initial population of the country and Y_0 be its initial food production. Let the average consumption be a unit. Therefore, food required initially aX_0. It is given

$$Y_p = aX_0\left(\frac{90}{100}\right) = 0.9\,aX_0 \qquad \ldots\text{(i)}$$

Let X be the population of the country in year t.

$$\text{Then,}\quad \frac{dX}{dt} = \text{Rate of change of population}$$
$$= \frac{3}{100}X = 0.03\,X$$

$$\Rightarrow \quad \frac{dX}{X} = 0.03\,dt \;\Rightarrow\; \int\frac{dX}{X} = \int 0.03\,dt$$

$$\Rightarrow \quad \log X = 0.03\,t + c$$

$$\Rightarrow \quad X = A\cdot e^{0.03\,t},\text{ where } A = e^c$$

At $t = 0, X = X_0$, thus $X_0 = A$

$$\therefore \quad X = X_0\,e^{0.03\,t}$$

Let Y be the food production in year t.

$$\text{Then,}\quad Y = Y_0\left(1+\frac{4}{100}\right)^t = 0.9aX_0\,(1.04)^t$$

$$\because \quad Y_0 = 0.9\,aX_0 \qquad \text{[from Eq. (i)]}$$

Food consumption in the year t is $aX_0\,e^{0.03\,t}$.

$$\text{Again,}\quad Y - X \geq 0 \qquad \text{[given]}$$

$$\Rightarrow \quad 0.9\,X_0\,a\,(1.04)^t > a\,X_0\,e^{0.03\,t}$$

$$\Rightarrow \quad \frac{(1.04)^t}{e^{0.03\,t}} > \frac{1}{0.9} = \frac{10}{9}.$$

Taking log on both sides, we get

$$t\,[\log\,(1.04) - 0.03] \geq \log 10 - \log 9$$

$$\Rightarrow \quad t \geq \frac{\log 10 - \log 9}{\log\,(1.04) - 0.03}$$

Thus, the least integral values of the year n, when the country becomes self-sufficient is the smallest integer greater than or equal to $\dfrac{\log 10 - \log 9}{\log\,(1.04) - 0.03}$.

Question 13

A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the X-axis. Determine the equation of the curve. **(1999, 10M)**

Solution:

Equation of normal at point (x, y) is

$$Y - y = -\frac{dx}{dy}(X - x) \quad \ldots(i)$$

Distance of perpendicular from the origin to Eq. (i)

$$= \frac{\left| y + \frac{dx}{dy} \cdot x \right|}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}}$$

Also, distance between P and X-axis is $|y|$.

$$\therefore \quad \frac{\left| y + \frac{dx}{dy} \cdot x \right|}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}} = |y|$$

$$\Rightarrow \quad y^2 + \frac{dx}{dy} \cdot x^2 + 2xy\frac{dx}{dy} = y^2\left[1 + \left(\frac{dx}{dy}\right)^2\right]$$

$$\Rightarrow \quad \left(\frac{dx}{dy}\right)^2 (x^2 - y^2) + 2xy\frac{dx}{dy} = 0$$

$$\Rightarrow \quad \frac{dx}{dy}\left[\left(\frac{dx}{dy}\right)(x^2 - y^2) + 2xy\right] = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = 0 \quad \text{or} \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

But $\frac{dx}{dy} = 0$

$\Rightarrow \; x = c$, where c is a constant.

Since, curve passes through (1, 1), we get the equation of the curve as $x = 1$.

The equation $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ is a homogeneous equation.

Put $\quad y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x^2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v} = -\frac{v^2 + 1}{2v}$$

$$\Rightarrow \quad \frac{-2v}{v^2 + 1}\,dv = \frac{dx}{x}$$

$$\Rightarrow \quad c_1 - \log(v^2 + 1) = \log|x|$$

$$\Rightarrow \quad \log|x|(v^2 + 1) = c_1 \quad \Rightarrow |x|\left(\frac{y^2}{x^2} + 1\right) = e^{c_1}$$

$\Rightarrow x^2 + y^2 = \pm e^{c_1} x$ or $x^2 + y^2 = \pm e^{c}x$ is passing through (1, 1).

$$\therefore \quad 1 + 1 = \pm e^{c} \cdot 1$$

$$\Rightarrow \quad \pm e^{c} = 2$$

Hence, required curve is $x^2 + y^2 = 2x$.

Question 14

A and B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at the time.

One hour after the water is released, the quantity of water in reservoir A is $1\frac{1}{2}$ times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water? **(1997, 7M)**

Solution:

$\frac{dV}{dt} \propto V$ for each reservoir.

$$\frac{dV}{dx} \propto -V_A \quad \Rightarrow \quad \frac{dV_A}{dt} = -K_1V_A$$

[K_1 is the proportional constant]

$$\Rightarrow \quad \int_{V_A}^{V'_A} \frac{dV_A}{V_A} = -K_1\int_0^t dt$$

$$\Rightarrow \quad \log\frac{V'_A}{V_A} = -K_1 t \quad \Rightarrow \quad V'_A = V_A \cdot e^{-K_1 t} \quad \ldots(i)$$

Similarly for B, $\quad V'_B = V_B \cdot e^{-K_2 t} \quad \ldots(ii)$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{V'_A}{V'_B} = \frac{V_A}{V_B} \cdot e^{-(K_1 - K_2)t}$$

It is given that at $t = 0, V_A = 2V_B$ and at

$$t = \frac{3}{2}, V'_A = \frac{3}{2}\,V'_B$$

Thus, $\frac{3}{2} = 2 \cdot e^{-(K_1 - K_2)t} \quad \Rightarrow e^{-(K_1 - K_2)} = \frac{3}{4} \quad \ldots(iii)$

Now, let at $t = t_0$ both the reservoirs have some quantity of water. Then,

$$V'_A = V'_B$$

From Eq. (iii), $2e^{-(K - K_2)} = 1$

$$\Rightarrow \quad 2 \cdot \left(\frac{3}{4}\right)^{t_0} = 1$$

$$t_0 = \log_{3/4}(1/2)$$

Question 15

Determine the equation of the curve passing through the origin in the form $y = f(x)$, which satisfies the differential equation $\frac{dy}{dx} = \sin(10x + 6y)$ **(1996, 5M)**

Solution:

Given, $\dfrac{dy}{dx} = \sin(10x + 6y)$

Let $10x + 6y = t$ ...(i)

$\Rightarrow 10 + 6\dfrac{dy}{dx} = \left(\dfrac{dt}{dx}\right)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{6}\left(\dfrac{dt}{dx} - 10\right)$

Now, the given differential equation becomes

$$\sin t = \frac{1}{6}\left(\frac{dt}{dx} - 10\right)$$

$\Rightarrow 6\sin t = \dfrac{dt}{dx} - 10$

$\Rightarrow \dfrac{dt}{dx} = 6\sin t + 10$

$\Rightarrow \dfrac{dt}{6\sin t + 10} = dx$

On integrating both sides, we get

$$\frac{1}{2}\int \frac{dt}{3\sin t + 5} = x + c \quad \text{...(ii)}$$

Let $I_1 = \displaystyle\int \frac{dt}{3\sin t + 5} = \int \frac{dt}{3\left(\dfrac{2\tan t/2}{1 + \tan^2 t/2}\right) + 5}$

$$= \int \frac{(1 + \tan^2 t/2)\,dt}{\left(6\tan\dfrac{t}{2} + 5 + 5\tan^2\dfrac{t}{2}\right)}$$

Put $\tan t/2 = u$

$\Rightarrow \dfrac{1}{2}\sec^2 t/2\,dt = du \Rightarrow dt = \dfrac{2\,du}{\sec^2 t/2}$

$\Rightarrow dt = \dfrac{2\,du}{1 + \tan^2 t/2} \quad \Rightarrow dt = \dfrac{2\,du}{1 + u^2}$

$\therefore I_1 = \displaystyle\int \frac{2(1 + u^2)du}{(1 + u^2)(5u^2 + 6u + 5)} = \frac{2}{5}\int \frac{du}{u^2 + \dfrac{6}{5}u + 1}$

$$= \frac{2}{5}\int \frac{du}{u^2 + \dfrac{6}{5}u + \dfrac{9}{25} - \dfrac{9}{25} + 1}$$

$$= \frac{2}{5}\int \frac{du}{\left(u + \dfrac{3}{5}\right)^2 + \left(\dfrac{4}{5}\right)^2} = \frac{2}{5}\cdot\frac{5}{4}\tan^{-1}\left(\frac{u + 3/5}{4/5}\right)$$

$$= \frac{1}{2}\tan^{-1}\left[\frac{5u + 3}{4}\right] = \frac{1}{2}\tan^{-1}\left[\frac{5\tan t/2 + 3}{4}\right]$$

On putting this in Eq. (ii), we get

$$\frac{1}{4}\tan^{-1}\left[\frac{5\tan\dfrac{t}{2} + 3}{4}\right] = x + c$$

$\Rightarrow \tan^{-1}\left[\dfrac{5\tan\dfrac{t}{2} + 3}{4}\right] = 4x + 4c$

$\Rightarrow \dfrac{1}{4}[5\tan(5x + 3y) + 3] = \tan(4x + 4c)$

$\Rightarrow 5\tan(5x + 3y) + 3 = 4\tan(4x + 4c)$

When $x = 0$, $y = 0$, we get

$$5\tan 0 + 3 = 4\tan(4c)$$

$\Rightarrow \dfrac{3}{4} = \tan 4c$

$\Rightarrow 4c = \tan^{-1}\dfrac{3}{4}$

Then, $5\tan(5x + 3y) + 3 = 4\tan\left(4x + \tan^{-1}\dfrac{3}{4}\right)$

$\Rightarrow \tan(5x + 3y) = \dfrac{4}{5}\tan\left(4x + \tan\dfrac{3}{4}\right) - \dfrac{3}{5}$

$\Rightarrow 5x + 3y = \tan^{-1}\left[\dfrac{4}{5}\left\{\tan\left(4x + \tan^{-1}\dfrac{3}{4}\right)\right\} - \dfrac{3}{5}\right]$

$\Rightarrow 3y = \tan^{-1}\left[\dfrac{4}{5}\left\{\tan\left(4x + \tan^{-1}\dfrac{3}{4}\right)\right\} - \dfrac{3}{5}\right] - 5x$

$\Rightarrow y = \dfrac{1}{3}\tan^{-1}\left[\dfrac{4}{5}\left\{\tan\left(4x + \tan^{-1}\dfrac{3}{4}\right)\right\} - \dfrac{3}{5}\right] - \dfrac{5x}{3}$