Q1: Let $f(x) = (\sin(\tan^{-1}x) + \sin(\cot^{-1}x))^2 - 1$, where $|x| > 1$. If $dy/dx = 1/2\ d/dx(\sin^{-1}(f(x)))$ and $y(\sqrt{3}) = \pi/6$, then $y(-\sqrt{3})$ is equal to

(a) $\pi/3$ (b) $2\pi/3$ (c) $-\pi/6$ (d) $\pi/7$

Solution:

$f(x) = (\sin(\tan^{-1}x) + \sin(\cot^{-1}x))^2 - 1$

Let $\tan^{-1}(x) = A$ where $A \in (-\pi/2, -\pi/4) \cup (\pi/4, \pi/2)$

$\Rightarrow (\sin(\tan^{-1}x) + \sin(\cot^{-1}x))^2 - 1 = (\sin A + \cos B)^2 - 1$

$= 1 + 2 \sin A \cos A - 1$

$= \sin 2A$

$= 2x/(1+x^2)$

Given that, $dy/dx = (1/2)\ d/dx(\sin^{-1}(f(x)))$

$\Rightarrow dy/dx = -1/(1+x^2)$ for $|x| > 1$

($x > 1$ and $x < -1$)

To find the value of $y(-\sqrt{3})$, integrate dy/dx. To integrate the expression, interval should be continuous. So we have to integrate the expression in both the intervals.

$\Rightarrow y = -\tan^{-1}x + c_1$ for $x>1$ and $y = -\tan^{-1}x + c_2$ for $x < -1$

For $x > 1$, $c_1 = \pi/2$ [because $y(\sqrt{3}) = \pi/6$]

But c_2 can’t be determined as no other information is given for $x < -1$, so can’t determine the value of c_2. Therefore, all the options can be true.

Q2: If $\alpha = 3 \sin^{-1}(6/11)$ and $\beta = 3 \cos^{-1}(4/9)$ where the inverse trig functions take only the principal values, then the right option is

(a) $\cos\beta > 0$ (b) $\cos(\alpha + \beta) > 0$ (c) $\sin \beta < 0$ (d) $\cos\alpha < 0$

Answer: (b), (c) and (d)

Solution:

$\alpha = 3 \sin^{-1}(6/11) > 3 \sin^{-1}(6/12) > 3 \sin^{-1}(1/2) > \pi / 2$

$\beta = 3 \cos^{-1}(4/9) > 3 \cos^{-1}(4/8) > 3 \cos^{-1}(1/2) > \pi$

$\alpha + \beta > 3\pi/2$

$\sin \beta < 0, \cos \alpha < 0, \cos (\alpha + \beta) > 0$

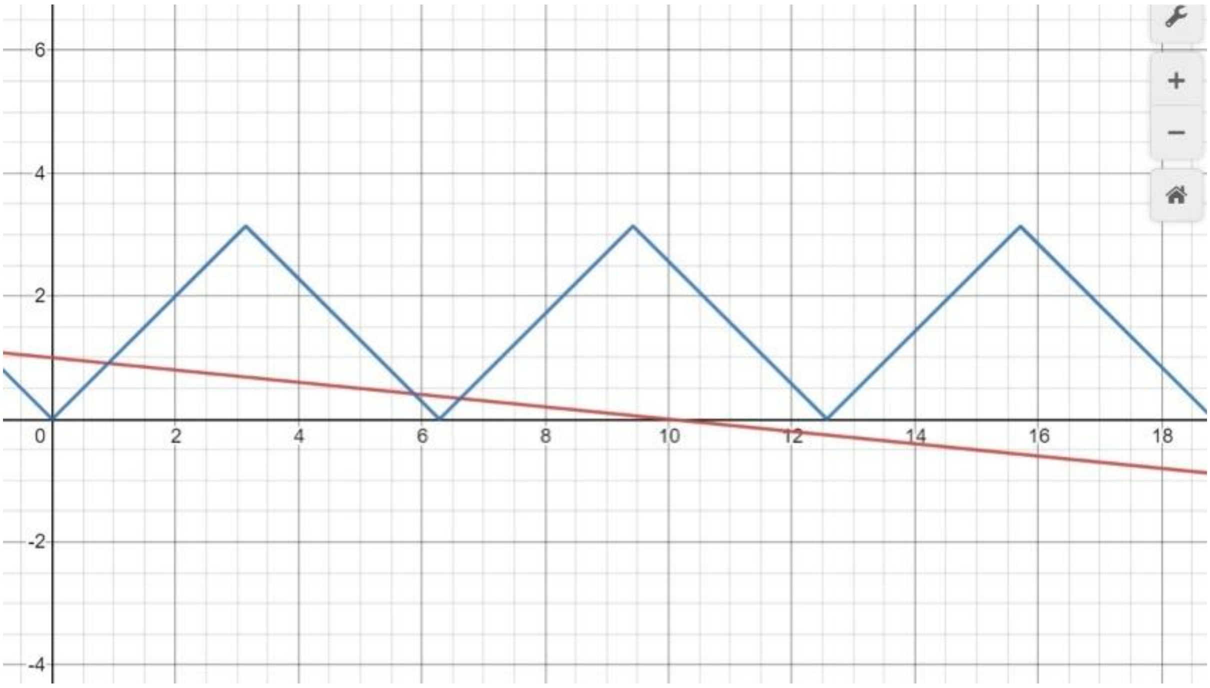
Q3: Let $f: [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation $f(x) = (10-x)/10$ is

(a) 1 (b) 2 (c) 3 (d) None of these

Answer: (c)

Solution:

Draw graph for $f(x) = \cos^{-1}(\cos x)$ and $f(x) = (10-x)/10$



Both the equations intersect at three different points, so the number of solutions is 3.

Q3: Let $f(x) = x \cos^{-1}(\sin(-|x|))$, $x \in (-\pi/2, \pi/2)$ then which of the following is true?

(a) $f'(0) = -\pi/2$

(b) f' is decreasing in $(-\pi/2, 0)$ and increasing in $(0, \pi/2)$

(c) f is not differentiable at $x = 0$

(d) f' is increasing in $(-\pi/2, 0)$ and decreasing in $(0, \pi/2)$

Answer: (b)

Solution:

$f(x) = x \cos^{-1}(\sin(-|x|))$ (Given)

=> $f(x) = x \cos^{-1}(-\sin(|x|))$

[As sine is an odd function]
=> $f(x) = x [\pi - \cos^{-1}(\sin(|x|))$

=> $f(x) = x [\pi - (\pi/2 - \sin^{-1}(\sin(|x|))$

=> $f(x) = x(\pi/2 + |x|)$

=> $f(x) = x(\pi/2 + x)$ when $x > 0$ and $f(x) = x(\pi/2 - x)$ when $x <= 0$

Therefore, $f'(x)$ is decreasing $(-\pi/2, 0)$ and increasing in $(0, \pi/2)$.