Linear programming

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1 Basics

Linear Programming deals with the problem of optimizing a linear *objective function* subject to linear equality and inequality *constraints* on the *decision variables*. Linear programming has many practical applications (in transportation, production planning, ...). It is also the building block for combinatorial optimization. One aspect of linear programming which is often forgotten is the fact that it is also a useful proof technique. In this first chapter, we describe some linear programming *formulations* for some classical problems. We also show that linear programs can be expressed in a variety of equivalent ways.

1.1 Formulations

1.1.1 The Diet Problem

In the diet model, a list of available foods is given together with the nutrient content and the cost per unit weight of each food. A certain amount of each nutrient is required per day. For example, here is the data corresponding to a civilization with just two types of grains (G1 and G2) and three types of nutrients (starch, proteins, vitamins):

	Starch	Proteins	Vitamins	Cost ($/kg)
G1	5	4	2	0.6
G2	7	2	1	0.35

Nutrient content and cost per kg of food.

The requirement per day of starch, proteins and vitamins is 8, 15 and 3 respectively. The problem is to find how much of each food to consume per day so as to get the required amount per day of each nutrient at minimal cost.

When trying to formulate a problem as a linear program, the first step is to decide which *decision variables* to use. These variables represent the unknowns in the problem. In the diet problem, a very natural choice of decision variables is:

- x_1: number of units of grain G1 to be consumed per day,
- x_2: number of units of grain G2 to be consumed per day.

The next step is to write down the *objective function*. The objective function is the function to be minimized or maximized. In this case, the objective is to minimize the total cost per day which is given by $z = 0.6x_1 + 0.35x_2$ (the value of the objective function is often denoted by z).

Finally, we need to describe the different *constraints* that need to be satisfied by x_1 and x_2. First of all, x_1 and x_2 must certainly satisfy $x_1 \geq 0$ and $x_2 \geq 0$. Only nonnegative amounts of food can be eaten! These constraints are referred to as *nonnegativity constraints*. Nonnegativity constraints appear in most linear programs. Moreover, not all possible values for x_1 and x_2 give rise to a diet with the required amounts of nutrients per day. The amount of starch in x_1 units of G1 and x_2 units of G2 is $5x_1 + 7x_2$ and this amount must be at least 8, the daily requirement of starch. Therefore, x_1 and x_2 must satisfy $5x_1 + 7x_2 \geq 8$. Similarly, the requirements on the amount of proteins and vitamins imply the constraints $4x_1 + 2x_2 \geq 15$ and $2x_1 + x_2 \geq 3$.

This diet problem can therefore be formulated by the following linear program:

$$\begin{array}{ll} \text{Minimize} & z = 0.6x_1 + 0.35x_2 \\ \text{subject to:} & \\ & 5x_1 + 7x_2 \geq 8 \\ & 4x_1 + 2x_2 \geq 15 \\ & 2x_1 + x_2 \geq 3 \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

Some more terminology. A *solution* $x = (x_1, x_2)$ is said to be *feasible* with respect to the above linear program if it satisfies all the above constraints. The set of feasible solutions is called the *feasible space* or *feasible region*. A feasible solution is *optimal* if its objective function value is equal to the smallest value z can take over the feasible region.

1.1.2 The Transportation Problem

Suppose a company manufacturing widgets has two factories located at cities F1 and F2 and three retail centers located at C1, C2 and C3. The monthly demand at the retail centers are (in thousands of widgets) 8, 5 and 2 respectively while the monthly supply at the factories are 6 and 9 respectively. Notice that the total supply equals the total demand. We are also given the cost of transportation of 1 widget between any factory and any retail center.

	C1	C2	C3
F1	5	5	3
F2	6	4	1

Cost of transportation (in 0.01$/widget).

In the *transportation problem*, the goal is to determine the quantity to be transported from each factory to each retail center so as to meet the demand at minimum total shipping cost.

In order to formulate this problem as a linear program, we first choose the decision variables. Let x_{ij} ($i = 1, 2$ and $j = 1, 2, 3$) be the number of widgets (in thousands) transported from factory Fi to city Cj. Given these x_{ij}'s, we can express the total shipping cost, i.e. the objective function to be minimized, by

$$5x_{11} + 5x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + x_{23}.$$

We now need to write down the constraints. First, we have the nonnegativity constraints saying that $x_{ij} \geq 0$ for $i = 1, 2$ and $j = 1, 2, 3$. Moreover, we have that the demand at each retail center must be met. This gives rise to the following constraints:

$$x_{11} + x_{21} = 8,$$

$$x_{12} + x_{22} = 5,$$

$$x_{13} + x_{23} = 2.$$

Finally, each factory cannot ship more than its supply, resulting in the following constraints:

$$x_{11} + x_{12} + x_{13} \leq 6,$$

$$x_{21} + x_{22} + x_{23} \leq 9.$$

These inequalities can be replaced by equalities since the total supply is equal to the total demand. A linear programming formulation of this transportation problem is therefore given by:

$$\begin{array}{ll} \text{Minimize} & 5x_{11} + 5x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + x_{23} \\ \text{subject to:} & \\ & x_{11} + x_{21} = 8 \\ & x_{12} + x_{22} = 5 \\ & x_{13} + x_{23} = 2 \\ & x_{11} + x_{12} + x_{13} = 6 \\ & x_{21} + x_{22} + x_{23} = 9 \\ & x_{11} \geq 0, x_{21} \geq 0, x_{31} \geq 0, \\ & x_{12} \geq 0, x_{22} \geq 0, x_{32} \geq 0. \end{array}$$

Among these 5 equality constraints, one is *redundant*, i.e. it is implied by the other constraints or, equivalently, it can be removed without modifying the feasible space. For example, by adding the first 3 equalities and substracting the fourth equality we obtain the last equality. Similarly, by adding the last 2 equalities and substracting the first two equalities we obtain the third one.

1.2 Representations of Linear Programs

A linear program can take many different forms. First, we have a minimization or a maximization problem depending on whether the objective function is to be minimized or maximized. The constraints can either be inequalities ($\leq$ or $\geq$) or equalities. Some variables might be unrestricted in sign (i.e. they can take positive or negative values; this is denoted by $\gtreqless 0$) while others might be restricted to be nonnegative. A general linear program in the decision variables $x_1, \ldots, x_n$ is therefore of the following form:

$$\begin{array}{ll} \text{Maximize or Minimize} & z = c_0 + c_1 x_1 + \ldots + c_n x_n \\ \text{subject to:} & \end{array}$$

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \begin{array}{c} \leq \\ \geq \\ = \end{array} b_i \qquad i = 1, \ldots, m$$

$$x_j \left\{ \begin{array}{l} \geq 0 \\ \gtreqless 0 \end{array} \right. \qquad j = 1, \ldots, n.$$

The problem data in this linear program consists of c_j $(j = 0, \ldots, n)$, b_i $(i = 1, \ldots, m)$ and a_{ij} $(i = 1, \ldots, m,\ j = 1, \ldots, n)$. c_j is referred to as the objective function coefficient of x_j or, more simply, the *cost coefficient* of x_j. b_i is known as the *right-hand-side* (RHS) of equation i. Notice that the constant term c_0 can be omitted without affecting the set of optimal solutions.

A linear program is said to be in *standard form* if

- it is a maximization program,
- there are only equalities (no inequalities) and
- all variables are restricted to be nonnegative.

In matrix form, a linear program in standard form can be written as:

$$\begin{aligned} \text{Max} \quad & z = c^T x \\ \text{subject to:} \quad & \\ & Ax = b \\ & x \geq 0. \end{aligned}$$

where

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

are column vectors, c^T denote the transpose of the vector c, and $A = [a_{ij}]$ is the $m \times n$ matrix whose $i, j-$element is a_{ij}.

Any linear program can in fact be transformed into an equivalent linear program in standard form. Indeed,

- If the objective function is to minimize $z = c_1x_1 + \ldots + c_nx_n$ then we can simply maximize $z' = -z = -c_1x_1 - \ldots - c_nx_n$.
- If we have an inequality constraint $a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i$ then we can transform it into an equality constraint by adding a *slack* variable, say s, restricted to be nonnegative: $a_{i1}x_1 + \ldots + a_{in}x_n + s = b_i$ and $s \geq 0$.
- Similarly, if we have an inequality constraint $a_{i1}x_1 + \ldots + a_{in}x_n \geq b_i$ then we can transform it into an equality constraint by adding a *surplus* variable, say s, restricted to be nonnegative: $a_{i1}x_1 + \ldots + a_{in}x_n - s = b_i$ and $s \geq 0$.
- If x_j is unrestricted in sign then we can introduce two new decision variables x_j^+ and x_j^- restricted to be nonnegative and replace every occurrence of x_j by $x_j^+ - x_j^-$.

For example, the linear program

$$\begin{aligned} \text{Minimize} \quad & z = 2x_1 - x_2 \\ \text{subject to:} \quad & \\ & x_1 + x_2 \geq 2 \\ & 3x_1 + 2x_2 \leq 4 \\ & x_1 + 2x_2 = 3 \\ & x_1 \gtrless 0, x_2 \geq 0. \end{aligned}$$

is equivalent to the linear program

$$\begin{aligned} \text{Maximize} \quad & z' = -2x_1^+ + 2x_1^- + x_2 \\ \text{subject to:} \quad & \\ & x_1^+ - x_1^- + x_2 - x_3 = 2 \\ & 3x_1^+ - 3x_1^- + 2x_2 + x_4 = 4 \\ & x_1^+ - x_1^- + 2x_2 = 3 \\ & x_1^+ \geq 0, x_1^- \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0. \end{aligned}$$

with decision variables $x_1^+, x_1^-, x_2, x_3, x_4$. Notice that we have introduced different slack or surplus variables into different constraints.

In some cases, another form of linear program is used. A linear program is in *canonical form* if it is of the form:

$$\begin{aligned} \text{Max} \quad & z = c^T x \\ \text{subject to:} \quad & \\ & Ax \leq b \\ & x \geq 0. \end{aligned}$$

A linear program in canonical form can be replaced by a linear program in standard form by just replacing $Ax \leq b$ by $Ax + Is = b$, $s \geq 0$ where s is a vector of slack variables and I is the $m \times m$ identity matrix. Similarly, a linear program in standard form can be replaced by a linear program in canonical form by replacing $Ax = b$ by $A'x \leq b'$ where $A' = \begin{bmatrix} A \\ -A \end{bmatrix}$ and $b' = \begin{pmatrix} b \\ -b \end{pmatrix}$.

2 The Simplex Method

In 1947, George B. Dantzig developed a technique to solve linear programs — this technique is referred to as the *simplex method.*

2.1 Brief Review of Some Linear Algebra

Two systems of equations $Ax = b$ and $\bar{A}x = \bar{b}$ are said to be equivalent if $\{x : Ax = b\} = \{x : \bar{A}x = \bar{b}\}$. Let E_i denote equation i of the system $Ax = b$, i.e. $a_{i1}x_1 + \ldots + a_{in}x_n = b_i$. Given a system $Ax = b$, an *elementary row operation* consists in replacing E_i either by αE_i where α is a *nonzero* scalar or by $E_i + \beta E_k$ for some $k \neq i$. Clearly, if $\bar{A}x = \bar{b}$ is obtained from $Ax = b$ by an elementary row operation then the two systems are equivalent. (Exercise: prove this.) Notice also that an elementary row operation is reversible.

Let a_{rs} be a nonzero element of A. A pivot on a_{rs} consists of performing the following sequence of elementary row operations:

- replacing E_r by $\bar{E}_r = \frac{1}{a_{rs}} E_r$,
- for $i = 1, \ldots, m$, $i \neq r$, replacing E_i by $\bar{E}_i = E_i - a_{is}\bar{E}_r = E_i - \frac{a_{is}}{a_{rs}} E_r$.

After pivoting on a_{rs}, all coefficients in column s are equal to 0 except the one in row r which is now equal to 1. Since a pivot consists of elementary row operations, the resulting system $\bar{A}x = \bar{b}$ is equivalent to the original system.

Elementary row operations and pivots can also be defined in terms of matrices. Let P be an $m \times m$ invertible (i.e. P^{-1} exists[1]) matrix. Then $\{x : Ax = b\} = \{x : PAx = Pb\}$. The two types of elementary row operations correspond to the matrices (the coefficients not represented are equal to 0):

$$P = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \alpha & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{matrix} \text{ and } P = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \beta & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow i \\ \\ \leftarrow k \\ \\ \\ \end{matrix} .$$

Pivoting on a_{rs} corresponds to premultiplying $Ax = b$ by

$$P = \begin{pmatrix} 1 & & & -a_{1s}/a_{rs} & & & \\ & \ddots & & & & & \\ & & 1 & -a_{r-1,s}/a_{rs} & & & \\ & & & 1/a_{rs} & & & \\ & & & -a_{r+1,s}/a_{rs} & 1 & & \\ & & & & & \ddots & \\ & & & -a_{ms}/a_{rs} & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow r. \\ \\ \\ \\ \end{matrix}$$

2.2 The Simplex Method on an Example

For simplicity, we shall assume that we have a linear program of (what seems to be) a rather special form (we shall see later on how to obtain such a form):

- the linear program is in standard form,
- $b \geq 0$,
- there exists a collection B of m variables called a *basis* such that
  - the submatrix A_B of A consisting of the columns of A corresponding to the variables in B is the $m \times m$ identity matrix and
  - the cost coefficients corresponding to the variables in B are all equal to 0.

For example, the following linear program has this required form:

[1]This is equivalent to saying that $\det P \neq 0$ or also that the system $Px = 0$ has $x = 0$ as unique solution

$$\begin{array}{lllllll} \text{Max } z = 10 + & 20\, x_1 & + & 16\, x_2 & + & 12\, x_3 & & & & \\ \text{subject to} & & & & & & & & & \\ & x_1 & & & & & + x_4 & & & = & 4 \\ & 2\, x_1 & + & x_2 & + & x_3 & & +x_5 & & = & 10 \\ & 2\, x_1 & + & 2x_2 & + & x_3 & & & + x_6 & = & 16 \\ & \multicolumn{6}{l}{x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.} \end{array}$$

In this example, $B = \{x_4, x_5, x_6\}$. The variables in B are called *basic* variables while the other variables are called *nonbasic*. The set of nonbasic variables is denoted by N. In the example, $N = \{x_1, x_2, x_3\}$.

The advantage of having $A_B = I$ is that we can quickly infer the values of the basic variables given the values of the nonbasic variables. For example, if we let $x_1 = 1, x_2 = 2, x_3 = 3$, we obtain

$$x_4 = 4 - x_1 = 3,$$

$$x_5 = 10 - 2x_1 - x_2 - x_3 = 3,$$

$$x_6 = 16 - 2x_1 - 2x_2 - x_3 = 7.$$

Also, we don't need to know the values of the basic variables to evaluate the cost of the solution. In this case, we have $z = 10 + 20x_1 + 16x_2 + 12x_3 = 98$. Notice that there is no guarantee that the so-constructed solution be feasible. For example, if we set $x_1 = 5, x_2 = 2, x_3 = 1$, we have that $x_4 = 4 - x_1 = -1$ does not satisfy the nonnegativity constraint $x_4 \geq 0$.

There is an assignment of values to the nonbasic variables that needs special consideration. By just letting all nonbasic variables to be equal to 0, we see that the values of the basic variables are just given by the right-hand-sides of the constraints and the cost of the resulting solution is just the constant term in the objective function. In our example, letting $x_1 = x_2 = x_3 = 0$, we obtain $x_4 = 4, x_5 = 10, x_6 = 16$ and $z = 10$. Such a solution is called a *basic feasible solution* or *bfs*. The feasibility of this solution comes from the fact that $b \geq 0$. Later, we shall see that, when solving a linear program, we can restrict our attention to basic feasible solutions. The simplex method is an iterative method that generates a sequence of basic feasible solutions (corresponding to different bases) and eventually stops when it has found an optimal basic feasible solution.

Instead of always writing explicitely these linear programs, we adopt what is known as the *tableau format*. First, in order to have the objective function play a similar role as the other constraints, we consider z to be a variable and the objective function as a constraint. Putting all variables on the same side of the equality sign, we obtain:

$$-z + 20x_1 + 16x_2 + 12x_3 = -10.$$

We also get rid of the variable names in the constraints to obtain the tableau format:

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	
1	20	16	12				-10
	1	0	0	1			4
	2	1	1		1		10
	2	2	1			1	16

Our bfs is currently $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 4, x_5 = 10, x_6 = 16$ and $z = 10$. Since the cost coefficient c_1 of x_1 is positive (namely, it is equal to 20), we notice that we can increase z by increasing x_1 and keeping x_2 and x_3 at the value 0. But in order to maintain feasibility, we must have that $x_4 = 4 - x_1 \geq 0$, $x_5 = 10 - 2x_1 \geq 0$, $x_6 = 16 - 2x_1 \geq 0$. This implies that $x_1 \leq 4$. Letting $x_1 = 4, x_2 = 0, x_3 = 0$, we obtain $x_4 = 0, x_5 = 2, x_6 = 8$ and $z = 90$. This solution is also a bfs and corresponds to the basis $B = \{x_1, x_5, x_6\}$. We say that x_1 has entered the basis and, as a result, x_4 has left the basis. We would like to emphasize that there is a *unique* basic solution associated with any basis. This (not necessarily feasible) solution is obtained by setting the nonbasic variables to zero and deducing the values of the basic variables from the m constraints.

Now we would like that our tableau reflects this change by showing the dependence of the new basic variables as a function of the nonbasic variables. This can be accomplished by *pivoting* on the element a_{11}. Why a_{11}? Well, we need to pivot on an element of column 1 because x_1 is entering the basis. Moreover, the choice of the row to pivot on is dictated by the variable which leaves the basis. In this case, x_4 is leaving the basis and the only 1 in column 4 is in row 1. After pivoting on a_{11}, we obtain the following tableau:

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	
1		16	12	-20			-90
	1	0	0	1			4
		1	1	-2	1		2
		2	1	-2		1	8

Notice that while pivoting we also modified the objective function row as if it was just like another constraint. We have now a linear program which is equivalent to the original one from which we can easily extract a (basic) feasible solution of value 90. Still z can be improved by increasing x_s for $s = 2$ or 3 since these variables have a positive cost coefficient[2] $\bar{c}_s$. Let us choose the one with the greatest $\bar{c}_s$; in our case x_2 will enter the basis. The maximum value that x_2 can take while x_3 and x_4 remain at the value 0 is dictated by the constraints $x_1 = 4 \geq 0$, $x_5 = 2 - x_2 \geq 0$ and $x_6 = 8 - 2x_2 \geq 0$. The tightest of these inequalities being $x_5 = 2 - x_2 \geq 0$, we have that x_5 will leave the basis. Therefore, pivoting on $\bar{a}_{22}$, we obtain the tableau:

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	
1			-4	12	-16		-122
	1		0	1	0		4
		1	1	-2	1		2
			-1	2	-2	1	4

The current basis is $B = \{x_1, x_2, x_6\}$ and its value is 122. Since $12 > 0$, we can improve the current basic feasible solution by having x_4 enter the basis. Instead of writing explicitely the constraints on x_4 to compute the level at which x_4 can enter the basis, we perform the *min ratio test*. If x_s is the variable that is entering the basis, we compute

$$\min_{i:\bar{a}_{is}>0} \{\bar{b}_i / \bar{a}_{is}\}.$$

The argument of the minimum gives the variable that is exiting the basis. In our example, we obtain $2 = \min\{4/1, 4/2\}$ and therefore variable x_6 which is the basic variable corresponding to row 3 leaves the basis. Moreover, in order to get the updated tableau, we need to pivot on $\bar{a}_{34}$. Doing so, we obtain:

[2] By simplicity, we always denote the data corresponding to the current tableau by $\bar{c}, \bar{A}$, and $\bar{b}$.

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	
1			2		-4	-6	-146
	1		1/2		1	-1/2	2
		1	0		-1	1	6
			-1/2	1	-1	1/2	2

Our current basic feasible solution is $x_1 = 2, x_2 = 6, x_3 = 0, x_4 = 2, x_5 = 0, x_6 = 0$ with value $z = 146$. By the way, why is this solution feasible? In other words, how do we know that the right-hand-sides (RHS) of the constraints are guaranteed to be nonnegative? Well, this follows from the min ratio test and the pivot operation. Indeed, when pivoting on $\bar{a}_{rs}$, we know that

- $\bar{a}_{rs} > 0$,
- $\frac{\bar{b}_r}{\bar{a}_{rs}} \leq \frac{\bar{b}_i}{\bar{a}_{is}}$ if $\bar{a}_{is} > 0$.

After pivoting the new RHS satisfy

- $\bar{b}_r = \frac{\bar{b}_r}{\bar{a}_{rs}} \geq 0$,
- $\bar{b}_i = \bar{b}_i - \frac{\bar{a}_{is}}{\bar{a}_{rs}} \geq \bar{b}_i \geq 0$ if $\bar{a}_{is} \leq 0$ and
- $\bar{b}_i = \bar{b}_i - \frac{\bar{a}_{is}}{\bar{a}_{rs}} = \bar{a}_{is}\left(\frac{\bar{b}_i}{\bar{a}_{is}} - \frac{\bar{b}_r}{\bar{a}_{rs}}\right) \geq 0$ if $\bar{a}_{is} > 0$.

We can also justify why the solution keeps improving. Indeed, when we pivot on $\bar{a}_{rs} > 0$, the constant term $\bar{c}_0$ in the objective function becomes $\bar{c}_0 + \bar{b}_r * \bar{c}_s/\bar{a}_{rs}$. If $\bar{b}_r > 0$, we have a strict improvement in the objective function value since by our choice of entering variable $\bar{c}_s > 0$. We shall deal with the case $\bar{b}_r = 0$ later on.

The bfs corresponding to $B = \{1, 2, 4\}$ is not optimal since there is still a positive cost coefficient. We see that x_3 can enter the basis and, since there is just one positive element in row 3, we have that x_1 leaves the basis. We thus pivot on $\bar{a}_{13}$ and obtain:

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	
1	-4				-8	-4	-154
	2		1		2	-1	4
	0	1			-1	1	6
	1			1	0	0	4

The current basis is $\{x_3, x_2, x_4\}$ and the associated bfs is $x_1 = 0, x_2 = 6, x_3 = 4, x_4 = 4, x_5 = 0, x_6 = 0$ with value $z = 154$. This bfs is *optimal* since the objective function reads $z = 154 - 4x_1 - 8x_5 - 4x_6$ and therefore cannot be more than 154 due to the nonnegativity constraints.

Through a sequence of pivots, the simplex method thus goes from one linear program to another equivalent linear program which is trivial to solve. Remember the crucial observation that a pivot operation does not alter the feasible region.

In the above example, we have not encountered several situations that may typically occur. First, in the min ratio test, several terms might produce the minimum. In that case, we can arbitrarily select one of them. For example, suppose the current tableau is:

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	
1		16	12	-20			-90
	1	0	0	1			4
		1	1	-2	1		**2**
		2	1	-2		1	**4**

and that x_2 is entering the basis. The min ratio test gives $2 = \min\{2/1, 4/2\}$ and, thus, either x_5 or x_6 can leave the basis. If we decide to have x_5 leave the basis, we pivot on $\bar{a}_{22}$; otherwise, we pivot on $\bar{a}_{32}$. Notice that, in any case, the pivot operation creates a zero coefficient among the RHS. For example, pivoting on $\bar{a}_{22}$, we obtain:

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	
1			-4	12	-16		-122
	1		0	1	0		4
		1	1	-2	1		2
			-1	2	-2	1	**0**

A bfs with $\bar{b}_i = 0$ for some i is called *degenerate*. A linear program is *nondegenerate* if *no* bfs is degenerate. Pivoting now on $\bar{a}_{34}$ we obtain:

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	
1			2		-4	-6	-122
	1		1/2		1	-1/2	4
		1	0		-1	1	2
			-1/2	1	-1	1/2	0

This pivot is degenerate. A pivot on $\bar{a}_{rs}$ is called *degenerate* if $\bar{b}_r = 0$. Notice that a degenerate pivot alters neither the $\bar{b}_i$'s nor $\bar{c}_0$. In the example, the bfs is $(4, 2, 0, 0, 0, 0)$ in both tableaus. We thus observe that several bases can correspond to the same basic feasible solution.

Another situation that may occur is when x_s is entering the basis, but $\bar{a}_{is} \leq 0$ for $i = 1, \ldots, m$. In this case, there is no term in the min ratio test. This means that, while keeping the other nonbasic variables at their zero level, x_s can take an arbitrarily large value without violating feasibility. Since $\bar{c}_s > 0$, this implies that z can be made arbitrarily large. In this case, the linear program is said to be *unbounded* or *unbounded from above* if we want to emphasize the fact that we are dealing with a maximization problem. For example, consider the following tableau:

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	
1		16	12	20			-90
	1	0	0	-1			4
		1	1	0	1		2
		2	1	-2		1	8

If x_4 enters the basis, we have that $x_1 = 4 + x_4$, $x_5 = 2$ and $x_6 = 8 + 2x_4$ and, as a result, for *any* nonnegative value of x_4, the solution $(4 + x_4, 0, 0, x_4, 2, 8 + 2x_4)$ is feasible and its objective function value is $90 + 20x_4$. There is thus no *finite optimum.*