(1) $\int \sin 4x \cdot e^{\tan^2 x} \cdot dx$

$= 2\int \sin 2x \cdot \cos 2x \cdot e^{\tan^2 x} dx = 2\int \frac{2\tan x}{1+\tan^2 x} \cdot \frac{1-\tan^2 x}{1+\tan^2 x} \cdot \frac{\sec^2 x}{(1+\tan^2 x)}$

$\tan^2 x = t \qquad 2\tan x \sec^2 x \cdot dx = dt$

$\therefore \cancel{e^{\tan^2 x} = t \Rightarrow 2\tan x}$

$= 2\int \frac{1-t}{(1+t)^3} e^t \cdot dt$

$= 2\int \left(\frac{-1-t+2}{(1+t)^3}\right) e^t$

$= 2\int \left(\frac{-1}{(1+t)^2} + \frac{2}{(1+t)^3}\right) e^t \cdot dt$

$\underbrace{\frac{-1}{(1+t)^2}}_{f(x)} \quad \underbrace{\frac{2}{(1+t)^3}}_{f'(x)}$

$= 2 \cdot \frac{-e^t}{(t+1)^2} = \boxed{\frac{-2e^t}{(t+1)^2} + c} \quad \boxed{t = \tan^2 x}$

Q. 2 $\int \frac{\sqrt{\cos 2x}}{\sin x} \cdot dx = -\ln\left|t + \sqrt{t^2-1}\right| + \sqrt{2}\ln\left|p + \sqrt{p^2 - \frac{1}{2}}\right| + c$

$t = \cot x \qquad p = \cos x$

$= \int \frac{\sqrt{\cos 2x}}{\sin x} = \int \sqrt{\operatorname{cosec}^2 x - 2} \cdot dx = \int \frac{\operatorname{cosec}^2 x - 2}{\sqrt{\operatorname{cosec}^2 x - 2}} \cdot dx = \frac{\operatorname{cosec}^2 x}{\sqrt{\cot^2 x - 1}} \quad \frac{\sqrt{2}\sin x}{\sqrt{\cos^2 x - \frac{1}{2}}}$

put $t = \cot x$ put $p = \cos x$

$= \int \frac{-dt}{\sqrt{t^2-1}} + \int \frac{\sqrt{2}\, dp}{\sqrt{t^2 - \frac{1}{2}}} = \boxed{-\ln\left|\cot x + \sqrt{\cot^2 x - 1}\right| + \sqrt{2}\ln\left|\cos x + \sqrt{\cos^2 x - \frac{1}{2}}\right| + c}$

(3) $\int \frac{5\tan x}{\tan x - 2}\,dx \Rightarrow x + 2\ln(\sin x - 2\cos x) + C$

$= \frac{5\sin x}{\sin x - 2\cos x} = 5\left[\frac{A(\sin x - 2\cos x) + B(2\sin x + \cos x)}{\sin x - 2\cos x}\right]$

$\therefore A + 2B = 1 \;\&\; B = 2A \quad \therefore A = \frac{1}{5} \;\&\; B = \frac{2}{5}$

$\therefore \int \frac{5\sin x}{\sin x - 2\cos x} = \int\left(1 + \frac{2(2\sin x + \cos x)}{\sin x - 2\cos x}\right).dx$

$= \boxed{x + 2\ln(\sin x - 2\cos x) + c}$

(4) $\int \csc^5 x\,.dx \longrightarrow \boxed{-\frac{1}{4}\csc^3 x\cot x + \frac{3}{4}\left(\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln|t+\sqrt{1+t^2}|\right)}$

$\csc x = t \qquad -\csc x\cot x\,.dx = dt \qquad t = \cot x$

$\therefore \int \csc^5 x\,.dx = \int -\frac{t^4}{\sqrt{t^2-1}}.dt$

$= -\left(\frac{t^5}{\sqrt{t^2-1}} - \int t.\frac{(4t^3(t^2-1) - 2t.t^4)}{t^2-1}.dt\right) = \frac{-t^5}{\sqrt{t^2-1}} + \int \frac{2t^6 - 2t^4 - 2t^4 + 2t^2 - 2t^2 + 2 - 2}{t^2-1}$

$= \boxed{\frac{-t^5}{\sqrt{t^2-1}} + \frac{2}{5}t^5 - \frac{2}{3}t^3 - 2t + \ln\frac{t+1}{t-1} + c} \qquad \boxed{t = \csc x}$

(5) $\int \sin^2 x.\cos^4 x\,.dx$

$= \int \frac{1}{16}(1 - \cos 4x)(\cos 2x + 1).dx = \int \frac{1}{16}(\cos 2x - \cos 4x\cos 2x - \cos 4x + 1)$

$= \int \frac{1}{16}\left(\cos 2x - \frac{\cos 6x}{2} - \frac{\cos 2x}{2} - \cos 4x + 1\right)$

$= \frac{1}{16}\left(\frac{\sin 2x}{2} - \frac{\sin 6x}{12} - \frac{\sin 2x}{4} - \frac{\sin 4x}{4} + x\right) + C$

⑥ $\int \sin^3 x \, (\cos x)^{\frac{1}{3}} \, dx$

⑦ $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) . dx$

⑧ $\int \frac{dx}{x + \sqrt{x^2+2x+2}}$

Ans 6 $\therefore \cos x = t^3 \quad \therefore -\sin x . dx = 3t^2 . dt$

$= -\int (1-t^6)\, t . 3t^2 . dt$

$= \int (3t^9 - 3t^3) . dt$

$= \boxed{\frac{3}{10} t^{10} - \frac{3}{4} t^4 + C}$ $\quad \boxed{t = (\cos x)^{\frac{1}{3}}}$

Ans 7 $\frac{2x+2}{\sqrt{(2x+2)^2+9}} = \sin t$, $(x+1) = \frac{3}{2} \tan\theta$

$dx = \frac{3}{2} \sec^2\theta . d\theta$

$= \int \sin^{-1} \sin\theta . \frac{3}{2} \sec^2\theta . d\theta$

$= \int \theta . \frac{3}{2} \sec^2\theta . d\theta = \frac{3}{2}\left(\theta . \tan\theta - \int \tan\theta . d\theta\right)$

$= \boxed{\frac{3}{2}\left(\theta \tan\theta + \ln \cos\theta\right) + C}$ $\quad \boxed{\theta = \tan^{-1} \frac{2}{3}(x+1)}$

Ans 8 $\displaystyle\int \frac{dx}{1+\sqrt{(x+1)^2+1}} = \int \frac{\sqrt{(x+1)^2+1}-1}{(x+1)^2}\cdot dx$

$\therefore$ put $(x+1) = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int \frac{(\sec\theta - 1)\sec^2\theta \cdot d\theta}{\tan^2\theta}$$

$$= \int \frac{(\sec\theta - 1)}{\sin^2\theta}\cdot d\theta = \int \frac{d\theta}{\sin^2\theta\cos\theta} + \cot\theta$$

put $\sin\theta = p$

$\cancel{\cos\theta}\cdot d\theta = \dfrac{dp}{\cos\theta}$

$$= \int \frac{dp}{p^2(1-p^2)}$$

$$= \int \frac{dp}{p^2} + \frac{dp}{1-p^2} = \cancel{-\frac{1}{p} + \int \frac{1}{2}\left(\frac{1}{1-p} + \frac{1}{1+p}\right)\cdot dp} = -\frac{1}{p} + \frac{1}{2}\ln\frac{1+p}{1-p}$$

$$= \cancel{\boxed{-\frac{1}{p} - \frac{1}{2}\ln\frac{p-1}{p+1}}} = \cancel{-\frac{1}{p} + \frac{1}{2}\ln(p\ p)}$$

Net integation

$$= \int \frac{dp}{p^2(1-p^2)} + \cot\theta$$

$$= \cancel{\boxed{-\frac{1}{p} + \frac{1}{2}\ln\frac{1-p^2}{p+p} + \cot\theta + C}} \quad \cancel{\boxed{\begin{array}{l} x+1 = \tan\theta \\ \sin\theta = p \end{array}}}$$

$$= -\cancel{\operatorname{cosec}\theta - \frac{1}{2}\ln\frac{\sin\theta - 1}{\sin\theta + 1}}$$

$$= \boxed{-\frac{1}{p} + \frac{1}{2}\ln\frac{1+p}{1-p} + \cot\theta + C} \quad \begin{array}{l} x+1 = \tan\theta \\ \sin\theta = p \end{array}$$

$$\boxed{-\operatorname{cosec}\theta + \frac{1}{2}\ln\frac{1+\sin\theta}{1-\sin\theta} + \cot\theta + C}$$ where $\boxed{\tan\theta = x+1}$

① $\displaystyle\int \frac{x^2+20 \, dx}{(x\sin x+5\cos x)^2} \Rightarrow -\cot\left(x+\tan^{-1}\frac{5}{x}\right)+c$

$$\cancel{\int \frac{dx}{\left(\frac{x\sin x}{\sqrt{x^2+25}}+\frac{5\cos x}{\sqrt{x^2+25}}\right)^2} = \int \frac{dx}{\sin^2\left(x+\sin^{-1}\frac{5}{\sqrt{x^2+25}}\right)}}$$

$$\cancel{\int \csc^2\left(x+\sin^{-1}\frac{5}{\sqrt{x^2+25}}\right) dx} \quad \frac{5 \cdot dx^2}{(x\sin x+5\cos x)^2}$$

$$= \int \frac{(x^2+20) \cdot dx}{(x^2+25)\left(\sin^2\left(x+\tan^{-1}\frac{5}{x}\right)\right)} \qquad \therefore x+\tan^{-1}\frac{5}{x} = t$$

$$= \int \csc^2 t \cdot dt \qquad 1+\frac{-\frac{5}{x^2}}{1+\left(\frac{5}{x}\right)^2} = dt$$

$$= \boxed{-\cot\left(x+\tan^{-1}\frac{5}{x}\right)+c} \qquad = \left(\frac{x^2+20}{x^2+25}\right) \cdot dx = dt$$

② $\sqrt{\sin 2x} \cdot \cos x \, dx$

$$\frac{1}{2}\int \sqrt{\sin 2x}\,(\cos x+\sin x)\,dx + \sqrt{\sin 2x}\,(\cos x-\sin x)\,dx$$

$$= \frac{1}{2}\int \sqrt{1-(\sin x-\cos x)^2}\,(\cos x+\sin x)\,dx - \sqrt{(\sin x+\cos x)^2-1}\,(\cos x-\sin x)\,dx$$

$$= \frac{1}{2}\int \sqrt{1-t^2}\,dt + \frac{1}{2}\int \sqrt{p^2-1}\,dp$$

$$= \frac{1}{2}\left(\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1} x\right) + \frac{1}{2}\left(\frac{p}{2}\sqrt{p^2-1}-\frac{1}{2}\ln\left(p+\sqrt{p^2-1}\right)\right)+c$$

$\boxed{t = \sin x-\cos x}$ $\quad$ $\boxed{p = \sin x+\cos x}$

(3) $\dfrac{e^{2x}-e^{x}+1}{(e^{x}\sin x+\cos x)(-e^{x}\cos x+\sin x)}$

$$=\int\frac{(e^{2x}-e^{x}+1)\,dx}{(e^{2x}+1)-\sin(x+\theta)\cos(x+\theta)}$$

$\theta=\tan^{-1}e^{-x}$

$$\int\frac{-2(e^{2x}-e^{x}+1)}{(e^{2x}+1)\cdot\sin(2x+2\theta)}$$

$2x+2\theta=t$

$2-\dfrac{2e^{-x}}{1+e^{-2x}}=dt$

$2\left(\dfrac{e^{2x}+1-e^{x}}{e^{2x}+1}\right)=dt$

$$=\int-\operatorname{cosec}t\,.dt$$

$$=\boxed{\ln(\operatorname{cosec}t+\cot t)+c}\quad\therefore\ \boxed{t=2x+2\tan^{-1}e^{-x}}$$

(4) $\displaystyle\int\frac{(\sin x)\,.dx}{\sin^3x+\cos^3x}$

$$=\frac{\sqrt{2}\sin x}{\left(\frac{\sin x+\cos x}{\sqrt2}\right)(2-\sin2x)}=\frac{\sqrt2\sin x}{\sin(x+\frac{\pi}{4})(2-\sin2x)}=\int\frac{\sqrt2\sin(t-\frac{\pi}{4})}{\sin t(2+\cos2t)}$$

$x+\dfrac{\pi}{4}=t$

$$=\frac{\sqrt2\left(\frac{\sin t}{\sqrt2}-\frac{\cot t}{\sqrt2}\right)\sqrt2}{\sin t(2+\cos2t)}=\frac{(\sin t-\cos t).dt}{\sin t(2+\cos2t)}=\int\frac{\sec^2t\,dt}{(3+\tan^2t)}-\frac{\cot\sec^2t\,dt}{(3+\tan^2t)}$$

$\therefore\ \tan t=p$

$\sec^2t.dt=dp$

$$=\int\frac{dp}{3+p^2}-\frac{dp}{p(3+p^2)}=\int\frac{dp}{3+p^2}-\frac{1}{3}\left(\frac{dp}{p}-\frac{p.dp}{3+p^2}\right)$$

$$=\cancel{\frac{1}{\sqrt3}\tan^{-1}\frac{\tan(x+\frac{\pi}{4})}{\sqrt3}}=\boxed{\frac{1}{\sqrt3}\tan^{-1}\left(\frac{\tan(x+\frac{\pi}{4})}{\sqrt3}\right)-\frac{1}{3}\ln\tan\left(x+\frac{\pi}{4}\right)+\frac{1}{6}\ln\left(3+\tan^2\left(x+\frac{\pi}{4}\right)\right)+c}$$

(5) $\displaystyle\int(\cos.2x)^{\frac{3}{2}}\cos x.dx$

(6) $\displaystyle\int\frac{\sin^2x.dx}{e^{2mx}(\cos x-m\sin x)^2}$

(7) $\displaystyle\int e^{x\sin x+\cos x}\left(\frac{x^4\cos^3x-x\sin x+\cos x}{x^2\cos^2x}\right).dx$

(8) $\int \frac{\operatorname{cosec}^2 x \cdot dx}{(\operatorname{cosec} x + \cot x)^{\frac{9}{2}}}$

Ans 5 $\int (\cos 2x)^{\frac{3}{2}} \cos x \cdot dx = \int (1-2\sin^2 x)^{\frac{3}{2}} \cdot \cos x \cdot dx$

$\sin x = t \qquad \cos x \cdot dx = dt$

$= \int (1-2t^2)^{\frac{3}{2}} \cdot dt = t(1-2t^2)^{\frac{3}{2}} - \frac{3}{2}\int t \cdot (1-2t^2)^{\frac{1}{2}} \cdot dt$

$1-2t^2 = p^2 \qquad \therefore -2t \cdot dt = 2p \cdot dp$

$= t(1-2t^2)^{\frac{3}{2}} + \frac{3}{4}\int p^2 \cdot dp = \boxed{t(1-2t^2)^{\frac{3}{2}} + \frac{p^3}{4} + C}$

$\boxed{t(1-2t^2)^{\frac{3}{2}} + \frac{(1-2t^2)^{\frac{3}{2}}}{4} + C} \quad \boxed{t = \sin x}$

Ans 6 $e^{-2mx}\left(\frac{-2m(f(x))}{\cos x - m\sin x} + \frac{f'(x)(\cos x - m\sin x) + f(x)(m\cos x + \sin x)}{(\cos x - m\sin x)^2}\right)$

$\therefore \frac{e^{-2mx}}{(\cos x - m\sin x)^2}\left(-2m(\cos x - m\sin x)f(x) + f(x)(m\cos x + \sin x) + f'(x)(\cos x - m\sin x)\right)$

$\frac{e^{-2mx}}{(\cos x - m\sin x)^2}\left(f(x)(-m\cos x + (2m^2+1)\sin x) + f'(x)(\cos x - m\sin x)\right)$

$\cos x + m\sin x$ to cancel this $\cos x$

$\therefore \frac{e^{-2mx}}{(\cos x - m\sin x)}\left((2m^2+m+1)\sin^2 x \cdot dx\right) \leftarrow$ for $f(x) = \cos x + m\sin x$

$\therefore \int \frac{e^{-2mx}}{(\cos x - m\sin x)^2}\sin^2 x \cdot dx = \boxed{\frac{e^{-2mx}}{(m^2+1)}\frac{(\cos x + m\sin x)}{(\cos x - m\sin x)}} + C$

Ans 7 $\int e^{x\sin x+\cos x}(x^2\cos x+1)+\int e^{x\sin x+\cos x}\left(\frac{x^2\cos^2 x-x\sin x+\cos x}{x^2\cos^2 x}\right)$

$= e^{x\sin x+\cos x}\cdot x - \int e^{x\sin x+\cos x}\left(\frac{x\cos x}{x\cos x}+\frac{x\sin x-\cos x}{x^2\cos^2 x}\right)$

as $\frac{d}{dx}(x\sin x+\cos x)=x\cos x$

$= e^{x\sin x+\cos x}\cdot x - e^{x\sin x+\cos x}\cdot\left(\frac{1}{x\cos x}\right)$

$=\boxed{e^{x\sin x+\cos x}\left(x-\frac{1}{x\cos x}\right)+c}$

Ans 8 $\int\frac{\operatorname{cosec}^2 x\,dx}{(\operatorname{cosec}x+\cot x)^{\frac{9}{2}}}=\int\frac{\sin x^{\frac{5}{2}}}{(1+\cos x)^{\frac{9}{2}}}=\int\frac{2^{\frac{5}{2}}\sin\frac{x}{2}^{\frac{5}{2}}\cos\frac{x}{2}^{\frac{5}{2}}}{2^{\frac{9}{2}}\cos^9\frac{x}{2}}$

$=\int\frac{1}{4}\left(\tan\frac{x}{2}\right)^{\frac{5}{2}}\sec^4\frac{x}{2}\,.dx$

$\tan\frac{x}{2}=t \qquad \frac{1}{2}\sec^2\frac{x}{2}.dx=dt$

$=\int\frac{1}{2}\frac{(1+t^2)}{t^{\frac{5}{2}}}\sec^2\frac{x}{2}.dx=\int\frac{1}{2}\frac{(1+t^2)}{t^{\frac{5}{2}}}.dt$

$=\frac{1}{2}\int\left(t^{-\frac{5}{2}}+t^{-\frac{1}{2}}\right).dt=\frac{1}{2}\left(-\frac{3}{2}t^{-\frac{3}{2}}+2t^{\frac{1}{2}}\right)+c$

$=\frac{1}{4}\left(\frac{t^2-3}{t^{\frac{3}{2}}}\right)+c \qquad t=\tan\frac{x}{2}$

$=\boxed{\frac{1}{4}\left(\frac{\tan^2\frac{x}{2}-3}{\left(\tan\frac{x}{2}\right)^{\frac{3}{2}}}\right)+c}$

$=\int\frac{1}{2}t^{\frac{5}{2}}(1+t^2).dt$

$=\frac{1}{2}\int\left(t^{\frac{5}{2}}+t^{\frac{9}{2}}\right).dt$

$=\frac{1}{2}\left(\frac{2}{7}\tan^{\frac{7}{2}}\frac{x}{2}+\frac{2}{11}\tan^{\frac{11}{2}}\frac{x}{2}\right)$

$=\frac{1}{7}\left(\tan\frac{x}{2}\right)^{\frac{7}{2}}+\frac{2}{11}\left(\tan\frac{x}{2}\right)^{\frac{11}{2}}+c$

Ans 6 (II) $\int \frac{\sin^2 x \, dx}{(e^{mx}(\cos x - m\sin x))^2}$ $\frac{d}{dx}(e^{mx}(\cos x - m\sin x)) = e^{mx}(-m^2-1)\sin x$

$$= \frac{-1}{m^2+1}\int \frac{\sin x}{e^{mx}} \frac{(-e^{mx}\cdot \sin x(m^2+1))\cdot dx}{(e^{mx}(\cos x - m\sin x))^2}$$

$$= \frac{1}{m^2+1}\left(\frac{e^{mx}\cos x - m e^{mx}\sin x}{e^{2mx}}\right)$$ By parts

$$= \frac{-1}{m^2+1}\left(\frac{\sin x}{e^{mx}}\left(\frac{1}{e^{mx}(\cos x - m\sin x)}\right)\right)$$

$$\frac{e^{-2mx}(\cos x + m\sin x)}{m^2+m+1}$$

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