Integration of Trigonometric function

Case I :- $\int \frac{dx}{a+b\cos^2 x}$, $\int \frac{dx}{a+b\sin^2 x}$, $\int \frac{dx}{a\cos^2 x+b\sin^2 x}$, $\int \frac{dx}{a\cos^2 x+b\sin^2 x+c}$

divide Nu & Deno by $\cos^2 x$ or $\sin^2 x$

Case II :- $\int \frac{dx}{a+b\cos x}$, $\int \frac{dx}{a+b\sin x}$, $\int \frac{dx}{a\cos x+b\sin x}$,

$\int \frac{dx}{a\cos x+b\sin x+c}$ put $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

Case III :- $\int \sin^n x \cdot \cos^m x$ $(m,n \in N)$

(i) If m, n are odd then $\sin x = t$ or $\cos x = t$
(ii) If m even, n odd then $\cos x = t$
(iii) If m odd, n even the put $\sin x = t$
(iv) If m, n are even then use trigonometric identities

Ex - $\int (\sin x)^4 (\cos x)^5 \cdot dx$

$= \int \sin^4 x \cdot \cos^4 x \cdot \cos x \cdot dx = \int \sin^4 x (1-\sin^2 x)^2 \cos x \cdot dx$

$= \int t^4 (1-t^2)^2 \cdot dt$

Ex $\int \sin^4 x \cos^6 x \, dx = \int \left(\frac{\sin 2x}{2}\right)^2 \times \left(1 + \frac{\cos 2x}{2}\right) dx$

$= \frac{1}{2^5} \int (\sin 2x \cdot \cos 2x + \sin^3 2x) \, dx$

$= \frac{1}{2^9} \int 2 \sin 4x (3 \sin 2x - \sin 6x) \, dx$

then use $2 \sin A \sin B$

Case-IV $\int \frac{(a \sin x + b \cos x + c)}{d \sin x + e \cos x + f} dx$

put $a \sin x + b \cos x + c = A(d \sin x + e \cos x + f) + B(\text{diff of } (d \sin x + e \cos x + f)) + C$

Case V. $\int \frac{(a \sin x + b \cos x)}{d \sin x + e \cos x} dx$

put $a \sin x + b \cos x = A(d \sin x + e \cos x) + B(\text{diff of } (d \sin x + e \cos x))$

Ex - $\int \frac{\sin x + 2\cos x + 3}{4 \sin x + 5 \cos x + 6} dx$

$\sin x + 2\cos x + 3 = A(4 \sin x + 5 \cos x + 6) + B(4 \cos x - 5 \sin x) + C$

$4A - 5B = 1 \rightarrow 16A - 20B = 4$ $\boxed{A = \frac{14}{41}}$

$5A + 4B = 2$ $25A + 20B = 10$

$6A + C = 3$

$$\boxed{\int A\left(\frac{4\sin x+5\cos x+6}{\text{Deno}}\right)+B\left(\frac{4\cos x-5\sin x}{\text{Deno}}\right)+\frac{C}{\text{Deno}}\,dx}$$

Case VI $\int\sqrt{a\sec^2x+b}$, $\int\sqrt{a\,\text{cosec}^2+b}\,.dx$, $\int\sqrt{a\tan^2x+b}$, $\int\sqrt{a\cot^2x+b}$

integrate by using rationalisation

Ex $\int\sqrt{a\sec^2x+b}\,.dx$

$$=\int\frac{a\sec^2x+b}{\sqrt{a\sec^2x+b}}=a\int\frac{\sec^2x\,.dx}{\sqrt{a+b+a\tan^2x}}+b\int\frac{b\,dx\cos x}{\sqrt{a+b\cos^2x}}$$

put $\tan x=t$ & $\sin x=p$

$$=a\int\frac{dt}{\sqrt{a+b+at^2}}+\int\frac{b\,dp}{\sqrt{a+b-bp^2}}$$

Case VII $\int f(\sin 2x)(\cos x-\sin x)\,dx$ or $\int f(\sin 2x)(\cos x+\sin x).dx$

here $\sin 2x=(\sin x+\cos x)^2-1$ $\qquad \sin 2x=1-(\sin x-\cos x)^2$

Case VIII $\int\frac{b+a\sin x}{(a+b\sin x)^2}.dx$ divide Nu & Deno. by $\cos^2x$

$\int\frac{b+a\cos x}{(a+b\cos x)^2}$ divide Nu & Deno by $\sin^2x$

Case IX $\int \tan^m x \cdot \sec^n x \, dx$

n even, put $\tan x = t$
n odd, put $\sec x = t$

Eg - $\int \tan^2 x \cdot \sec^3 x \, dx$

$\Rightarrow \int \sec x \cdot \tan x \sqrt{\sec^2 x - 1} \cdot \sec^2 x \, dx$

$\sec x = t$

$\int t^2 \sqrt{t^2 - 1} \, dt \Rightarrow \int (t^2-1)^{\frac{3}{2}} - \int \sqrt{t^2-1} \, dt$

$- \int (t^2-1)^{\frac{3}{2}} \, dt = \int \tan^3 \theta \cdot \sec\theta \cdot \tan\theta \, d\theta$

$= \int (\sec^5\theta + \sec\theta - 2\sec^3\theta) \, d\theta$

$\hookrightarrow 2\int \sqrt{1+\tan^2\theta} \cdot \sec^2\theta \, d\theta$

$I = \int \sec^3\theta \cdot \sec^2\theta \, d\theta$
(I) (II)

$t = \sec\theta$

$I = \sec^3\theta \cdot \tan\theta - \int 3\sec^2\theta \cdot \tan^2\theta$

$I = \sec^3\theta \cdot \tan\theta - 3\int \sec^3\theta (\sec^2\theta - 1) \, d\theta$

then taking $\int \sec^5\theta \, d\theta = I$
and solving.

$$\Rightarrow I = \int \underset{II}{1} \cdot \underset{I}{(t^2-1)^{\frac{3}{2}}} \cdot dt$$

$$I = (t^2-1)^{\frac{3}{2}} t - \frac{3}{2} \int (t^2-1)^{\frac{1}{2}} 2t \cdot t \cdot dt$$

$$\therefore I = t(t^2-1)^{\frac{3}{2}} - 3I - 3\int \sqrt{t^2-1} \cdot dt$$

$$4I = t(t^2-1)^{\frac{3}{2}} - 3\int \sqrt{t^2-1} \cdot dt$$

$$\therefore \int \tan^2 x \cdot \sec^3 x \cdot dx = I - \sqrt{t^2-1} \cdot dt$$

$$= \boxed{\cancel{t}\, t(t^2-1)^{\frac{3}{2}} - \frac{7}{4}\int \sqrt{t^2-1} + C}$$

M-2 $\int \frac{(t^2-1)^2}{\sqrt{t^2-1}} \cdot dt = \int \frac{t^4 - 2t^2 + 1}{\sqrt{t^2-1}} = \int \frac{t^4}{\sqrt{t^2-1}} - \int \frac{2t^2 - 2 + 1}{\sqrt{t^2-1}}$

can be done

$\int \frac{t^4}{\sqrt{t^2-1}} \rightsquigarrow t = \sec\theta$

$$\int \frac{t^6 \cdot dt}{t^3\sqrt{1-\frac{1}{t^2}}} = \int \frac{\cancel{\rho\rho}\, d\rho}{\rho} \left(\frac{1}{1-\rho^2}\right)^3$$

M-3 $\int \tan^2 x \cdot \sec^3 x \cdot dx$ | $\tan^2 x = t$

$2\tan x \cdot \sec^2 x \cdot dx = dt$

$$\int \frac{t\sqrt{1+t}}{2\sqrt{t}} = \frac{1}{2}\int \sqrt{\left(t+\frac{1}{2}\right)^2 - \frac{1}{4}} \cdot dt$$