Methods of integration

1. Substitution
2. Integration by parts
3. Trigonometric
4. Algebra
5. Partial fraction
6. Reduction formula.

1. Integration by substitution

Case-I :- $\int f(x).f'(x).dx$, $\int \frac{f'(x)}{f(x)}.dx$, $\int (f(x))^n f'(x).dx$

, $\int \frac{f'(x).dx}{(f(x))^n}$

→ We put $f(x) = t$, diff both side with respect to x

$f'(x).dx = dt \leftarrow f'(x) = \frac{dt}{dx}$

$\Rightarrow \int t.dt$, $\int \frac{dt}{t}$, $\int t^n.dt$, $\int \frac{dt}{t^n}$

$\Rightarrow \frac{t^2}{2} + C$ $\quad \ln t + C$

$= \frac{1}{2}(f(x))^2 + C$, $\ln f(x) + C$

Eg. $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$ put $\sin x = t$ $= \int \frac{dt}{t} = \ln \sin x + c$

$\int \sec x \, dx = \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)} = \int \frac{dt}{t} = \ln (\sec x + \tan x) + c$

$\boxed{t = (\sec x + \tan x)}$

<u>Case II</u> :- $\int g(f(x)) \cdot f'(x) \cdot dx$, $\int \frac{f'(x) \cdot dx}{g(f(x))}$, $\int \frac{f'(x) \cdot dx}{g(f'(x))}$

Eg $\int \cos(e^x) \cdot e^x \cdot dx$ $e^x = t$

$\int \cos t \cdot dt = \boxed{\sin e^x + c}$

<u>Case III</u> :- $\int \frac{f'(x)}{a + (f(x))^n}$ a is constant

Eg $\int \frac{\sqrt{x} \cdot dx}{1 + x^3} = \int \frac{\sqrt{x}}{1 + (x^{\frac{3}{2}})^2} dx$: put $x^{3/2} = t$

$= \int \frac{2 \cdot dt}{3(1 + t^2)} = \frac{2}{3} \tan^{-1} t = \boxed{\frac{2}{3} \tan^{-1} x^{\frac{3}{2}} + c}$

Case IV :- $\int \sqrt{a^2 - x^2} \cdot dx$, $\int \frac{dx}{\sqrt{a^2 - x^2}}$, $\int \frac{dx}{a^2 - x^2}$

Put $x = a \sin\theta$ or $x = a \cos\theta$

$x = a \sin\theta \Rightarrow dx = a \cos\theta \cdot d\theta$

$\int \frac{a \cos\theta \cdot d\theta}{\sqrt{a^2 - a^2 \sin^2\theta}} = \int \frac{a \cos\theta \cdot d\theta}{a \cos\theta} = \theta + c = \boxed{\sin^{-1}\left(\frac{x}{a}\right) + c}$

$$\int \frac{dx}{a^2-x^2} = -\int \frac{dx}{(x-a)(x+a)} = \frac{-1}{2a}\int \frac{(x+a)-(x-a)}{(x+a)(x-a)}\cdot dx$$

→ We are only talking about one of the methods and their may be other method possible.

If ⇒ $\int \sqrt{a^2-(cx+d)^2}\cdot dx$, $\int \frac{dx}{\sqrt{a^2-(cx+d)^2}}$ → $\int \frac{dx}{a^2-(cx+d)^2}$

Put : $cx+d = a\sin\theta$ or $cx+d = a\cos\theta$

Case V. $\int \sqrt{a^2+x^2}\cdot dx$, $\int \frac{dx}{\sqrt{a^2+x^2}}$, $\int \frac{dx}{a^2+x^2}$

Put $x = a\tan\theta$ or $x = a\cot\theta$

If → $\int \sqrt{a^2+(cx+d)^2}\cdot dx$, $\int \frac{dx}{\sqrt{a^2+(cx+d)^2}}$, $\int \frac{dx}{a^2+(cx+d)^2}$

Put $cx+d = a\tan\theta$ or $cx+d = a\cot\theta$

Case VI $\int \sqrt{x^2-a^2}\cdot dx$, $\int \frac{dx}{\sqrt{x^2-a^2}}$, $\int \frac{dx}{x^2-a^2}$

Put $x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$

Case VII : $\int \sqrt{\frac{a-x}{a+x}}\cdot dx$ (i) Put $x = a\cos 2\theta$

(ii) $\int \frac{\sqrt{a-x}}{\sqrt{a+x}} \times \sqrt{\frac{a-x}{a-x}}\cdot dx = \int \frac{a-x}{\sqrt{a^2-x^2}}\cdot dx$

$= \int \frac{a\,dx}{\sqrt{a^2-x^2}} - \int \frac{x\cdot dx}{\sqrt{a^2-x^2}}$

(iii) Put $\frac{a-x}{a+x} = t^2$

* If numerator & denominator are algebraic functions then take out max power of x, then it may change into $f(x)$ & $f'(x)$ most of the times

* If $\int \sqrt{f(x)}\,.dx$, then one method can be put $f(x) = t^2$ then solve

$$\frac{a-x}{a+x} = \frac{t^2}{1} \Rightarrow \frac{2a}{-2x} = \frac{t^2+1}{t^2-1} \qquad \frac{x}{a} = -\frac{(t^2-1)}{(t^2+1)}$$

$$\frac{dx}{a} = -\frac{((t^2+1)\,2t - (t^2-1)\,2t)\,.dt}{(t^2+1)^2}$$

$$dx = \frac{-4at\,.dx}{(t^2+1)^2} \qquad \text{Its integration from case VIII}$$

$$Ex = \int t\,\frac{-4at}{(t^2+1)^2}\,.dt = -4a\int\frac{t^2}{(t^2+1)^2}\,.dt$$

Case VIII Algebraic twins here k is constant

$$\int\frac{x^2}{x^4+kx^2+1}\,.dx,\quad \int\frac{dx}{x^4+kx^2+1},\quad \int\frac{x^2\,dx}{x^4+1},\quad \int\frac{dx}{x^4+1}$$

$$\int\frac{x^2}{x^4+kx^2+1}\,.dx = \frac{1}{2}\int\frac{2x^2}{x^4+kx^2+1}\,.dx = \frac{1}{2}\int\frac{(x^2+1)+(x^2-1)}{x^4+kx^2+1}\,.dx$$

$$= \frac{1}{2}\int\frac{(x^2+1)}{x^4+kx^2+1}\,.dx + \frac{1}{2}\int\frac{x^2-1}{x^4+kx^2+1}\,.dx$$

$$= \frac{1}{2}\int\frac{1+\frac{1}{x^2}}{(x-\frac{1}{x})^2+k+2} + \frac{1}{2}\int\frac{(1-\frac{1}{x^2})\,.dx}{(x+\frac{1}{x})^2+k-2}$$

$$Eg = \int\frac{5x^4+4x^5}{(x^5+x+1)^2} = \int\frac{x^5}{x^{10}}\frac{(\frac{5}{x}+4)}{(1+\frac{1}{x^4}+\frac{1}{x^5})^2} = \int\frac{\frac{1}{x^5}(\frac{5}{x}+4)\,.dx}{(1+\frac{1}{x^4}+\frac{1}{x^5})^2}$$

$$\int \frac{\frac{4}{x^5} + \frac{5}{x^6}}{\left(1 + \frac{1}{x^4} + \frac{1}{x^5}\right)^2} \, dx \qquad \frac{4}{x^5} + \frac{5}{x^6} \, dx = dt$$

$$= -\int \frac{dt}{t^2} = \frac{1}{t} + C = \frac{1}{1 + \frac{1}{x^4} + \frac{1}{x^5}} + C$$

Case-I - $\int \sqrt{\tan x}\,.dx$, $\int \sqrt{\cot x}\,.dx$

Put $\tan x = t^2$, put $\cot x = t^2$

Case II - $\int (x \pm \sqrt{x^2 \pm a^2})^n \,.dx$

Put $x \pm \sqrt{x^2 \pm a^2} = t$

Eg- $\int (x + \sqrt{x^2+a^2})^n \,.dx$

$x + \sqrt{x^2+a^2} = t$

$x^2 + a^2 = x^2 + t^2 - 2xt$

$x = \frac{t^2 - a^2}{2t}$

$dx = \frac{1}{2}\left(1 + \frac{a^2}{t^2}\right).dt$

$\boxed{= \frac{1}{2}\int t^n \left(1 + \frac{a^2}{t^2}\right).dt}$

Case III $\int \sqrt{(x-a)(b-x)}\,.dx$, $\int \sqrt{\frac{(x-a)}{(b-x)}}\,.dx$

for both

(I) Put $x = a\cos^2\theta + b\sin^2\theta$

(II) $\int \sqrt{(x-2)(4-x)} = \int \sqrt{4x - x^2 - 8 + 2x} = \int \sqrt{-(x^2 - 6x + 8)}$

$= \int \sqrt{1 - (x-3)^2}\,.dx$

(I) Put $\frac{x-a}{b-x} = t^2$

(II) $\int \frac{(x-a).dx}{\sqrt{(x-a)(b-x)}}$

Case-IV $\int \sqrt{(x-a)(x-b)}\, dx$, $\int \sqrt{\frac{x-a}{x-b}}\, dx$

(I) Put $x = a\sec^2\theta - b\tan^2\theta$

for both

(II) Making perfect square

Q.1 $\int (x^{3m} + x^{2m} + x^{m})(2x^{2m} + 3x^{m} + 6)^{\frac{1}{m}}\, dx$.

$\int (x^{3m-1} + x^{2m-1} + x^{m-1})(x^{m})^{\frac{1}{m}}(2x^{2m} + 3x^{m} + 6)^{\frac{1}{m}}\, dx$

$\int (x^{3m-1} + x^{2m-1} + x^{m-1})(\underbrace{2x^{3m} + 3x^{2m} + 6x^{m}}_{t})^{\frac{1}{m}}\, dx$

$6m(x^{3m-1} + x^{2m-1} + x^{m-1})dx = dt$

$\int \frac{dt}{6m} t^{\frac{1}{m}} = \frac{1}{6(m+1)} t^{\frac{m+1}{m}} = \boxed{\frac{(2x^{3m} + 3x^{2m} + 6x^{m})^{\frac{m+1}{m}}}{6(m+1)} + c}$

Q.2 $\frac{dx}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

Q.3 $\frac{dx}{\sin^3 x + \cos^3 x}$

Q.4 $\int \frac{dx}{\sqrt{1-\tan^2 x}}$

Ans3 M-II

$\frac{2\, dx}{(\sin x + \cos x)(2 - \sin 2x)} = \frac{\sqrt{2}\, dx}{\sin(x+\frac{\pi}{4})(2 - \sin 2x)}$

$x + \frac{\pi}{4} = t \quad dx = dt$

$= \sqrt{2} \int \frac{dt}{\sin t\,(2 - \sin(2t - \frac{\pi}{2}))} = \sqrt{2} \int \frac{dt}{\sin t\,(2+\cos 2t)}$

$= \sqrt{2} \int \frac{\sin t}{(1-\cos^2 t)}$