Illustration 27: Find the area bounded by the curve g(x), the x-axis and the lines at y = −1 and y = 4, where g(x) is the inverse of the function $f(x)=\frac{x^3}{24}+\frac{x^2}{8}+\frac{13x}{12}+1$. **(JEE MAIN)**

Sol: Here f(x) is a strictly increasing function therefore required area will be

$$A=\int_0^2(4-f(x))dx+\int_{-2}^0(f(x)+1)dx$$

Given $f(x)=\frac{x^3}{24}+\frac{x^2}{8}+\frac{13x}{12}+1$

$\Rightarrow$ f(0) = 1; f(2) = 4 and f(−2) = −1

Also, $f'(x)=\frac{x^2}{8}+\frac{x}{4}+\frac{13}{12}$,

i.e. f(x) is a strictly increasing function.

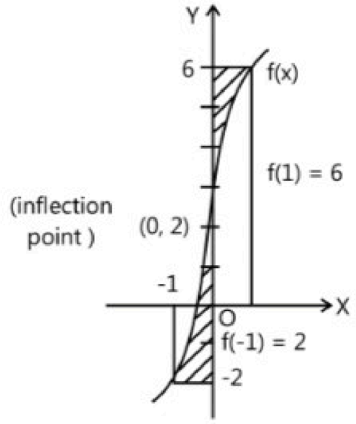


Figure 25.34

$$\therefore A=\int_0^2(4-f(x))dx+\int_{-2}^0(f(x)+1)dx$$

$$A=\int_0^2\left(4-\frac{x^3}{24}-\frac{x^2}{8}-\frac{13x}{12}-1\right)dx+\int_{-2}^0\left(\frac{x^3}{24}+\frac{x^2}{8}+\frac{13x}{12}+1+1\right)dx$$

$$\therefore A=\left[\left(3.2-\frac{2^4}{24.4}-\frac{2^3}{8.3}-\frac{13.2^2}{12.2}\right)-(0)\right]+\left[(0)-\left(\frac{2^4}{24.4}-\frac{2^3}{8.3}+\frac{13.2^2}{12.2}-2.2\right)\right]=\frac{16}{3}$$

Illustration 28: Let f(x) = x^3 + 3x + 2 and g(x) is the inverse of it. Find the area bounded by g(x), the x-axis and the ordinate at x = −2 and x = 6. **(JEE ADVANCED)**

Sol: Let $A=\int_{-2}^{6}\left|f^{-1}(x)\right|dx$

Substitute $x=f(u)$ or $u=f^{-1}(x)$

$$=\int_{f^{-1}(2)}^{f^{-1}(6)}|u|\,f^{-1}(u)\,du$$

$$=\int_{f^{-1}(2)}^{f^{-1}(6)}|4|\left(3u^2+3\right)du$$

We have, $f(-1)=2$ and $f(1)=6$

$$=\int_{-1}^{1}|u|\left(3u^2+3\right)du=2\int_0^1\left(3u^3+3u\right)du$$

$$=\left[\frac{3}{2}u^4+3u^2\right]_0^1=\frac{9}{2}\text{ Sq. units.}$$

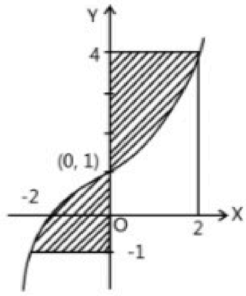


Figure 25.35

Illustration 31: Let C_1 & C_2 be the graphs of the function $y = x^2$ & $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graphs of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 & C_3 at Q & R respectively (see figure). If for every position of P(on C_1), the area of the shaded regions OPQ & ORP are equal, determine the function f(x). **(JEE ADVANCED)**

Sol: Similar to the above mentioned method.

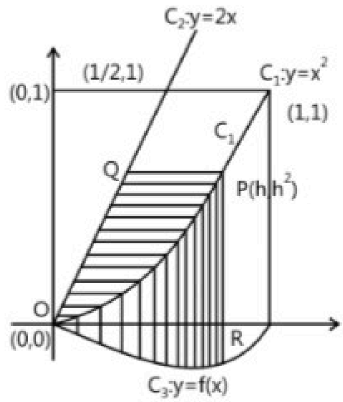


Figure 25.37

$$\int_0^{h^2}\left(\sqrt{y}-\frac{y}{2}\right)dy = \int_0^h (x^2 - f(x))\,dx$$ differentiate both sides w.r.t. h

$$\left(h-\frac{h^2}{2}\right)2h = h^2 - f(h)$$

$$f(h) = h^2 - \left(h-\frac{h^2}{2}\right)2h$$

$$= h^2 - h(2h - h^2) = h^2 - 2h^2 + h^3$$

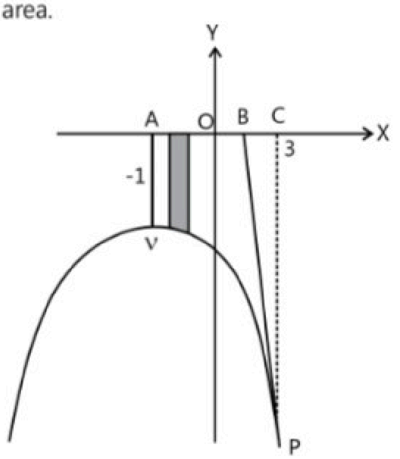
$$f(h) = h^3 - h^2$$

$$f(x) = x^3 - x^2 = x^2(x-1)$$

JEE Advanced/Boards

Example 1: A tangent is drawn to $x^2 + 2x - 4ky + 3 = 0$ at a point whose abscissa is 3. The tangent is perpendicular to $x + 3 = 2y$. Find the area bounded by the curve, this tangent, x-axis and line $x = -1$

Sol: As we know multiplication of slopes of two perpendicular line is – 1, by using this, we can obtain the value of k and will get standard equation. After that using integration with respective limit, we will be get required area.



$x^2 + 2x - 4ky + 3 = 0;\ \frac{dy}{dx} = \frac{x+1}{2k}$ Tangent is perpendicular to $x + 3 = 2y$

$\therefore\ \frac{x+1}{2k}\left(\frac{1}{2}\right) = -1$ at $x = 3$

$\Rightarrow 1/k = -1 \Rightarrow k = -1$

$\therefore$ Curve becomes $(x + 1)^2 = -4(y + 1/2)$ which is a parabola with vertex at $V(-1, -1/2)$.

Coordinates of P are $(3, -9/2)$.

Equation of tangent at P is $y + 9/2 = -2(x - 3)$

B is $(3/4, 0)$, C is $(3, 0)$

Required Area = Area (ACPV) – Area of triangle BPC.

$$= \left|\int_{-1}^{3} \frac{x^2+2x+3}{-4} dx\right| - \frac{1}{2}(BC)(CP)$$

$$= \frac{1}{4}\left|\left(\frac{x^3}{3} + x^2 + 3x\right)_{-1}^{3}\right| - \frac{1}{2}\left(3 - \frac{3}{4}\right)\left(\frac{9}{2}\right)$$

$$= \frac{1}{4}\left(27 + \frac{1}{3} - (1-3)\right) - \frac{81}{16} = \frac{109}{48} \text{ sq. units.}$$

Example 2: Let A_n be the area bounded by the curve $y = (\tan x)^n : n \in N$ and the lines $x = 0, y = 0$ and $x = \frac{\pi}{4}$. Prove that for $n > 2$, $A_n + A_{n-2} = \frac{1}{n-1}$ and deduce $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$.

Sol: We can write $(\tan x)^n$ as $\tan^{n-2} x\,(\sec^2 x - 1)$. Therefore by solving $A_n = \int_0^{\pi/4} \tan^{n-2} x\,(\sec^2 x - 1)dx$ we can prove given equation.

$$A_n = \int_0^{\pi/4} \tan^n x\, dx : n > 2$$

$$= \int_0^{\pi/4} \tan^{n-2} x\,(\sec^2 x - 1)dx$$

$$\text{or } A_n = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/2} - A_{n-2}$$

$$\therefore A_n + A_{n-2} = \frac{1}{n-1} \quad \text{... (i)}$$

$\tan^n x \le \tan^{n-2} x$

(as $0 \le \tan x \le 1$ for $0 \le x \le \frac{\pi}{4}$)

$\Rightarrow A_n < A_{n-2}$

$\therefore A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1}$ by (1)

$$\therefore A_n < \frac{1}{2(n-1)} \quad \text{... (ii)}$$

Similarly $A_{n+2} < A_n$

$\Rightarrow A_{n+2} + A_n < A_n + A_n$

or $\frac{1}{(n+2)-1} < 2A_n$ by (1)

$$\Rightarrow \frac{1}{2n+2} < A_n \quad \text{... (iii)}$$

$$\Rightarrow \frac{1}{2n+2} < A_n < \frac{1}{2n-2}$$

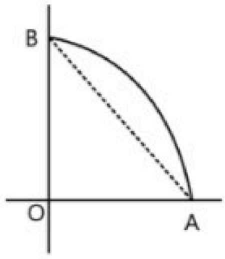
Example 3: A(a, 0) and B(0, b) are points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the area between the arc AB and chord AB of the ellipse is $\frac{1}{4}$ ab $(\pi - 2)$.

Sol: Area between the chord and ellipse = Area bounded by curve AB - Area of Δ OAB.

Equation of line AB is : $y = -\frac{b}{a}(x - a)$

Equation of curve AB is $y = \frac{b}{a}\sqrt{a^2 - x^2}$

Area of bounded region is



$$\int_0^a \left[\frac{b}{a}\sqrt{a^2 - x^2} - \left(-\frac{b}{a}(x-a)\right)\right] dx$$

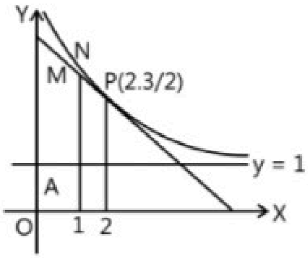
$$= \frac{b}{a}\left[0 + \frac{a^2\pi}{4} - \frac{a^2}{2}\right] = \frac{(\pi - 2)}{4}ab$$

Alternate method:

Area between the chord and ellipse = Area bounded by curve AB - Area of Δ OAB

$$= \frac{1}{4}\pi ab - \frac{1}{2}ab = \frac{(\pi - 2)ab}{4} \text{ sq. units}$$

Example 4: Find the area of the region bounded $y = \frac{1}{x} + 1, x = 1$ and tangent drawn at the point P(2, 3/2) to the curve $y = \frac{1}{x} + 1$.



Sol: Here first obtain equation of tangent and then use the formula for area.

Equation of tangent at P(2, 3/2) to $y = \frac{1}{x} + 1$ is $y - \frac{3}{2} = -\frac{1}{4}(x-2)$ or $x + 4y = 8$.

Required area is area of region PMN

$$\text{Area} = \int_1^2 \left(\left(\frac{1}{x} + 1\right) - \frac{8-x}{4}\right) dx$$

$$= \left(\ln x - x + \frac{1}{4}\frac{x^2}{2}\right)_1^2 = \ln 2 - \frac{5}{8} \text{ sq. units}$$

Example 5: Find the area of the region bounded by the x-axis and the curve $y = \frac{1}{2}(2 - 3x - 2x^2)$.

Sol: Here the curve will intersect the x-axis when $y = 0$, therefore by substituting $y = 0$ in the above equation we will get the points of intersection of curve and x – axis.

$\Rightarrow 2 - 3x - 2x^2 = 0$ or $(2 + x)(1 - 2x) = 0$ or $x = -2$, $x = \frac{1}{2}$

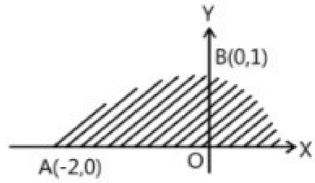
Thus, the curve passes through the points (–2, 0) and $\left(\frac{1}{2}, 0\right)$ on the x-axis.

It will have a turning points where $\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{1}{2}(-3 - 4x) = 0$ Þ$x = -\frac{3}{4}$

Also $\frac{d^2y}{dx^2} = -4$. That is, it is a max. at $x = \frac{3}{4}$

Also it cuts y-axis where $x = 0$, then $y = 1$. Thus the shape of the curve is as shown in the figure.



The required area is ABC. It is given by

$$\int_{-2}^{1/2} y\,dx = \int_{-2}^{1/2} \frac{1}{2}(2 - 3x - 2x^2)dx$$

$$= \frac{1}{2}\left[2x - \frac{3}{2}x^2 - \frac{2x^3}{3}\right]_{-2}^{1/2}$$

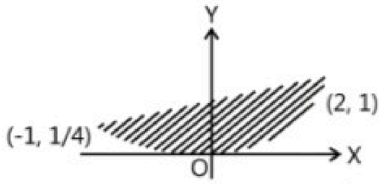
$$= \frac{1}{2}\left[2\left(\frac{1}{2}\right) - \frac{3}{2}\left(\frac{1}{2}\right)^2 - \frac{2}{3}\left(\frac{1}{2}\right)^3\right] -$$

$$\frac{1}{2}\left[2(-2) - \frac{3}{2}(-2)^2 - \frac{2}{3}(-2^3)\right]$$

$$= \frac{1}{2}\left(\frac{13}{24}\right) - \frac{1}{2}\left(-\frac{14}{3}\right) = \frac{125}{48} \text{ sq. units.}$$

Example 6: Find the area of the region bounded by the curve $x^2 = 4y$ and $x = 4y - 2$.

Sol: Solving given equation simultaneously, we will get the point of intersection. Using these points as the limits of integration, we calculate the required area.



The curve intersect each other, where $\frac{x^2}{4} = \frac{x+2}{4}$, or $x^2 - x - 2 = 0$, or $x = -1, 2$

Hence, the points of intersection are (–1, 1/4) and (2, 1). The region is plotted in figure. Since, the straight line $x = 4y - 2$ is always above the parabola $x^2 = 4y$ in the interval [–1, 2], the required area is given by

$$\text{Area} = \int_{-1}^{2} [f(x) - g(x)]dx$$

$$\text{Area} = \int_{-1}^{2}\left[\frac{x+2}{4}-\frac{x^2}{4}\right]dx = \frac{1}{4}\left[\frac{1}{2}x^2+2x-\frac{1}{3}x^3\right]_{-1}^{2}$$

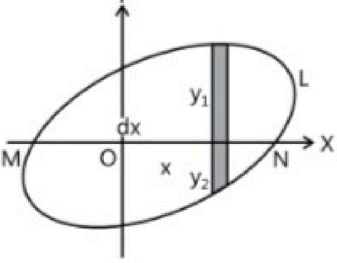
$$=\frac{1}{4}\left[\left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)\right] = \frac{9}{8} \text{ sq. units.}$$

Example 7: Find by using integration, the area of the ellipse $ax^2 + 2hxy + by^2 = 1$.

Sol: The equation can be put in the form $by^2 + 2hxy + (ax^2 - 1) = 0$

Cut an elementary strip.

Let the thickness of strip = dx



If y_1, y_2 be the values of y corresponding to any value at x.

Length of strip = $y_1 - y_2$

$$= \frac{2}{b}\sqrt{h^2x^2 - b(ax^2-1)} = \frac{2}{b}\sqrt{b-(ab-h^2)x^2}$$

$ab - h^2$ being positive here, since the conic is an ellipse.

The extreme values of x, are given by

$$y_1 - y_2 = 0, \text{ i.e., } x = \pm\sqrt{\frac{b}{ab-h^2}}$$

$$\text{Hence, the area required} = \int_{-\sqrt{b/ab-h^2)}}^{\sqrt{b/ab-h^2)}} (y_1 - y_2)dx$$

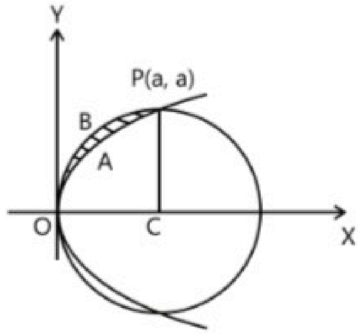
$$= \int_{-\sqrt{b/(ab-h^2)}}^{\sqrt{b/(ab-h^2)}} \frac{2}{b}\sqrt{b-(ab-h^2)x^2}\, dx$$

and putting $\sqrt{(ab-h^2)}\,x = \sqrt{b}\sin\theta$, this becomes

$$\frac{2}{\sqrt{(ab-h^2)}}\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta = \frac{\pi}{(ab-h^2)} \text{ sq. units.}$$

Example 8: Find the area of region lying above x-axis, and included between the circle $x^2 + y^2 = 2ax$ and the parabola $y^2 = ax$.

Sol: By solving these two equation simultaneously, we can obtain their intersection points and then by subtracting area of parabola from area of circle we will be get the result.



Solving the two equation, simultaneously we see that the two curves intersect at (0, 0), (a, a) and (a, –a). We have to find the area of the region OAPBO, where P is the point of intersection (a, a)

$$\text{Required area} = \int_0^a [f(x) - g(x)]dx$$

$$= \int_0^a \sqrt{(2ax - x^2)}\, dx - \int_0^a \sqrt{ax}\, dx$$

$$\text{Now, } \int_0^a \sqrt{(2ax-x^2)}\, dx = \int_0^a \sqrt{[a^2 - (a-x)^2]}\, dx$$

To evaluate this integral, we substitute $a - x = a \sin\theta$ and obtain

$$\int_0^a \sqrt{(2ax-x^2)}dx = \int_{\pi/2}^{0} (a\cos\theta)(-a\cos\theta)d\theta$$

$$= \int_0^{\pi/2} a^2\cos^2\theta\, d\theta = a^2\frac{1}{2}\frac{\pi}{2} = \frac{\pi a^2}{4}$$

$$\text{Also } \int_0^a \sqrt{(ax)}\, dx = \left|\sqrt{a}\frac{2}{3}x^{3/2}\right|_0^a = \frac{2a^2}{3}$$

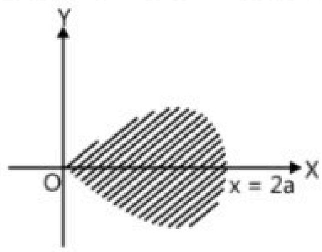
$$\therefore \text{ Required area} = a^2\left(\frac{\pi}{4}-\frac{2}{3}\right) \text{ sq. units}$$

Example 9: Prove that the area of the region bounded by the curve $a^4y^2 = x^5(2a - x)$, is $\frac{5}{4}$ times to that of the circle whose radius is a.

Sol: The curve is a loop lying between the line x = 0 and x = 2a and is symmetrical about the x-axis. Therefore the required area

$$= 2\int_0^{2a} y\, dx$$

$$= \frac{2}{a^2}\int_0^{2a} x^{5/2}\sqrt{2a-x}dx$$



To evaluate this integral, we put x = 2a sin²q. When, x = 0, θ = 0 and when x = 2a, $\theta = \frac{1}{2}\pi$ ∴ Required area

$$= \frac{2}{a^2}\int_0^{\pi/2}(2a)^{5/2}\sin^5\theta.\sqrt{2a}\cos\theta.4a\sin\theta\cos\theta\, d\theta$$

$$= 64a^2\int_0^{\pi/2}\sin^6\theta\cos^2\theta\, d\theta = 64a^2\,\frac{5.3.1.1}{8.6.4.2}\cdot\frac{\pi}{2} = \frac{5a^2\pi}{4}$$

$= \frac{5}{4}$x area of the circle whose radius is a.

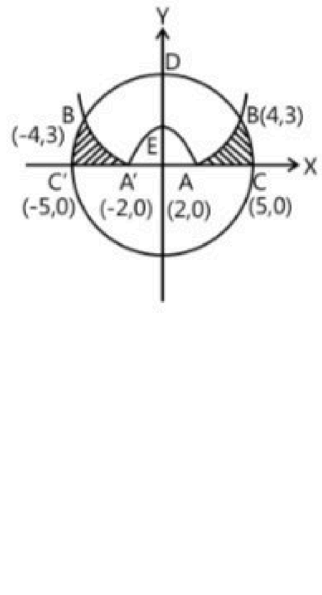
Example 10: Find the area bounded by the curves $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and $y = 0$.

Sol: Here $x^2 + y^2 = 25$ represent circle with centre at origin and radius 5 unit. Therefore the required area = 2 area ABC

$$= 2\left[\int_2^4 \frac{1}{4}(4 - x^2)dx + \int_4^5 \sqrt{25^2 - x^2}\, dx\right]$$

Note: Here the portion is also bounded by two curves but we do not apply $A = \int[f(x) - g(x)]dx$ rule.

Reason: Range of integration of both the curves is not same.



$$= 2\left[\int_2^4 \frac{1}{4}(x^2 - 4)dx + \int_4^5 \sqrt{(5^2 - x^2)}\, dx\right]$$

$$= \frac{2}{4}\left[\left(\frac{x^3}{3} - 4x\right)\right]_2^4 + 2\left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right]_4^5$$

$$= \frac{1}{2}\left[\left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)\right] +$$

$$2\left[\left(0 + \frac{25}{2}\sin^{-1}1\right) - \left(6 + \frac{25}{2}\sin^{-1}\frac{4}{5}\right)\right]$$

$$= \frac{1}{2}\left[\frac{32}{3}\right] + 25\sin^{-1}1 - 12 - 25\sin^{-1}\frac{4}{5}$$

$$= \left(\frac{16}{3} - 12\right) + 25\frac{\pi}{2} - 25\sin^{-1}\frac{4}{5}$$

$$= -\frac{20}{3} + 25\left(\frac{\pi}{2} - \sin^{-1}\frac{4}{5}\right) = \left(25\cos^{-1}\frac{4}{5} - \frac{20}{3}\right) \text{ sq. units.}$$