3.3 Area of a Curve in Parametric Form

If the given curve is in parametric form say x = f(t), y = g(t), then the area bounded by the curve with x-axis is equal to $\int_a^b y\,dx = \int_{t_2}^{t_1} g(t)f'(t)\,dt$ $\left[\because dx = d(f(t)) = f'(t)dt\right]$ Where t_1 and t_2 are the values of t corresponding to the values of a and b of x.

Illustration 9: Find the area bounded by the curve x = a cost, y = b sint in the first quadrant. **(JEE MAIN)**

Sol: Solve it using formula of area of a curve in parametric form.

The given equation is the parametric equation of ellipse, on simplifying we get $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$\therefore$ Required area $= \int_0^a y\,dx = \int_{\pi/2}^{0} (b\sin t(-a\sin t)dt) = ab\int_0^{\pi/2} \sin^2 t\,dt = \left(\frac{\pi ab}{4}\right)$.

3.4 Symmetrical Area

If the curve is symmetrical about a line or origin, then we find the area of one symmetrical portion and multiply it by the number of symmetrical portions to get the required area.

Illustration 10: Find the area bounded by the parabola $y^2 = 4x$ and its latus rectum. **(JEE MAIN)**

Sol: Here the given parabola is symmetrical about x – axis.

Hence required area $= 2\int_0^1 y\,dx$.

Since the curve is symmetrical about x-axis,

$\therefore$ The required Area $= 2\int_0^1 y\,dx = 2\int_0^1 \sqrt{4x}\,dx = 4\cdot\frac{2}{3}\left[x^{3/2}\right]_0^1 = \frac{8}{3}$

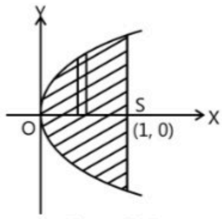

Figure 25.9

3.5 Positive and Negative Area

The area of a plane figure is always taken to be positive. If some part of the area lies above x-axis and some part lies below x-axis, then the area of two parts should be calculated separately and then add the numerical values to get the desired area.

If the curve crosses the x-axis at c (see Fig. 25.10), then the area bounded by the curve y = f(x) and the ordinates x = a and x = b, (b > a) is given by

$A = \left|\int_a^c f(x)dx\right| + \left|\int_c^b f(x)dx\right|$; $\quad A = \int_a^c f(x)dx - \int_c^b f(x)\,dx$

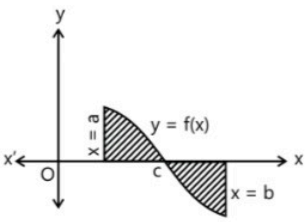

Figure 25.10

PLANCESS CONCEPTS

To reduce confusion of using correct sign for the components, take modulus and add all the absolute values of the components.

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Illustration 11: Find the area between the curve y = cos x and x-axis when π/4 < x < π **(JEE MAIN)**

Sol: Here some part of the required area lies above x-axis and some part lies below x-axis,. Hence by using above mentioned method we can obtain required area.

$\therefore$ Required area $= \int_{\pi/4}^{\pi/2}\cos x\,dx + \left|\int_{\pi/2}^{\pi}\cos x\,dx\right|$

$= [\sin x]_{\pi/4}^{\pi/2} + |[\sin x]_{\pi/2}^{\pi}| = \left(1 - 1/\sqrt{2}\right) + |0-1| = \frac{2\sqrt{2}-1}{\sqrt{2}}$

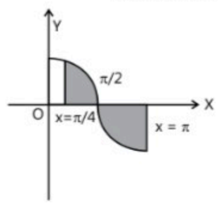


Figure 25.11

Illustration 12: Using integration, find the area of the triangle ABC, whose vertices are A (4, 1), B (6, 6) and C (8, 4) **(JEE ADVANCED)**

Sol: Here by using slope point form we can obtain respective equation of line by which given triangle is made. And after that by using integration method we can obtain required area.

Equation of line AB: $y - 1 = \frac{5}{2}(x-4) \Rightarrow y = \frac{5x}{2} - 9$

Equation of line AC: $y - 1 = \left(\frac{3}{4}\right)(x-4) \Rightarrow y = \frac{3x}{4} - 2$

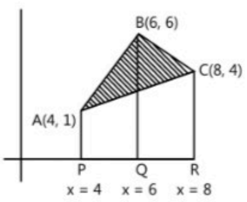


Figure 25.12

Equation of line BC: $(y-6) = \left(\frac{-2}{2}\right)(x-6) \Rightarrow y = -x + 12$

$\therefore$ The required area = Area of trapezium ABQP + Area of trapezium BCRQ – Area of trapezium ACRP

$$= \int_4^6 \left(\frac{5}{2}x - 9\right)dx + \int_6^8 (-x+12)dx - \int_4^8 \left(\frac{3}{4}x - 2\right)dx$$

$$= \left(\frac{5}{4}x^2 - 9x\right)_4^6 + \left(12x - \frac{x^2}{2}\right)_6^8 - \left(\frac{3}{8}x^2 - 2x\right)_4^8 = 7 + 10 - 10 = 7 \text{ sq. units.}$$

3.6 Area between Two Curves

(a) Area enclosed between two curves.

If $y = f_1(x)$ and $y = f_2(x)$ are two curves (where $f_1(x) > f_2(x)$), which intersect at two points, A ($x = a$) and B($x = b$), then the area enclosed by the two curves between A and B is

Common area $= \int_a^b (y_1 - y_2)dx = \int_a^b [f_1(x) - f_2(x)]dx$

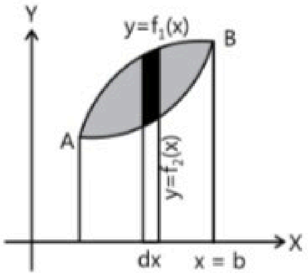


Figure 25.13

Illustration 13: Find the area between two curves $y^2 = 4ax$ and $x^2 = 4ay$. **(JEE MAIN)**

Sol: By using above mentioned formula of finding the area enclosed between two curves, we can obtain required area.

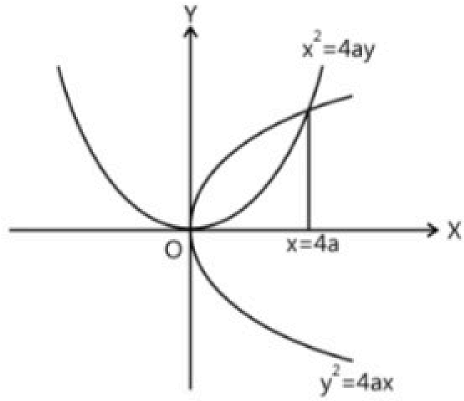


Figure 25.14

Given, $y^2 = 4ax$ ... (i)

$x^2 = 4ay$ ... (ii)

Solving (i) and (ii), we get $x = 4a$ and $y = 4a$.

So required area $= \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a}\right)dx = \left(2\sqrt{a}\frac{x^{3/2}}{3/2} - \frac{x^3}{12a}\right)_0^{4a}$

$$= \frac{4\sqrt{a}}{3}|4a|^{3/2} - \frac{64a^3}{12a} = \frac{16}{3}a^2$$

(b) Area enclosed by two curves intersecting at one point and the X-axis.

If $y = f_1(x)$ and $y = f_2(x)$ are two curves which intersect at a point P (α, β) and meet x-axis at A (a, 0) and B (b, 0) respectively, then the area enclosed between the curves and x-axis is given by

$$\text{Area} = \int_a^{\alpha} f_1(x)dx + \int_{\alpha}^{b} f_2(x)dx$$

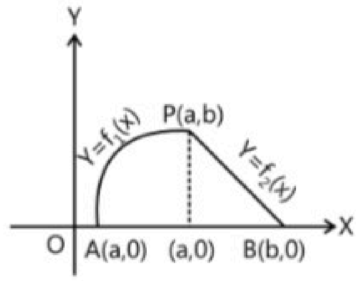


Figure 25.15

(c) Area bounded by two intersecting curves and lines parallel to y-axis.

The area bounded by two curves y = f(x) and y = g(x) (where $a \le x \le b$), when they intersect at $x = c \in (a, b)$, is given by $A = \int_a^b |f(x) - g(x)|\,dx \Rightarrow A = \int_a^c (f(x) - g(x))\,dx + \int_c^b (g(x) - f(x))\,dx$

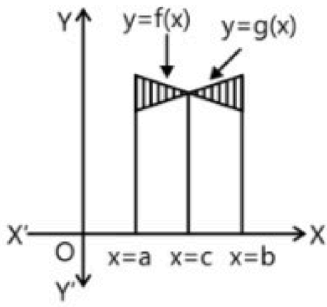


Figure 25.16

Illustration 14: Draw a rough sketch of the region enclosed between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$. Using method of integration, find the area of this enclosed region **(JEE ADVANCED)**

Sol: By solving given equations simultaneously, we will be get intersection points of circles and then by using integration method we can obtain required area.

The figure shown alongside is the sketch of the circles

$x^2 + y^2 = 4$ ... (i)

and, $(x - 2)^2 + y^2 = 4$ ... (ii)

From (i) and (ii), we have $(x - 2)^2 - x^2 = 0$

$\Rightarrow (x - 2 - x)(x - 2 + x) = 0 \quad \Rightarrow x = 1$ ... (iii)

Solving (i) and (iii), we get $y = \pm\sqrt{3}$

Therefore, the circles (i) and (ii) intersect at $A(1, \sqrt{3})$ and $B(1, -\sqrt{3})$.

Area of enclosed region = Area OACBO = 2 Area OACO

= 2 [Area OAD + Area ACD]

$= 2\int_0^1 \sqrt{4-(x-2)^2}\,dx + 2\int_1^2 \sqrt{4-x^2}\,dx$

$= 2\int_1^2 \sqrt{4-x^2}\,dx + 2\int_0^1 \sqrt{4-(x-2)^2}\,dx$

$$= 2\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2 + 2\left[\frac{(x-2)\sqrt{4-(x-2)^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 \quad \left[\because \sqrt{a^2-x^2}\,dx \Rightarrow \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= 2\left(\pi - \frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{6}\right)\right) + 2\left(-\frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{6}\right) + \pi\right) = \frac{8\pi}{3} - 2\sqrt{3} \text{ sq. units}$$

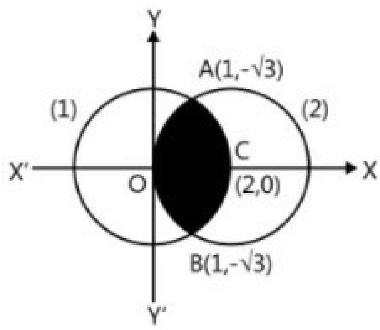
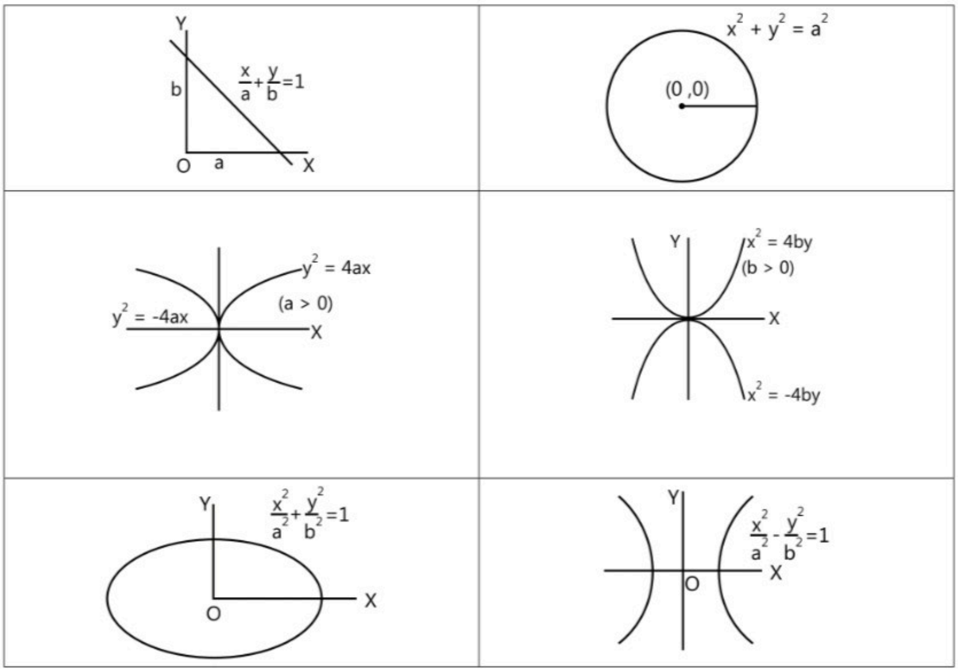


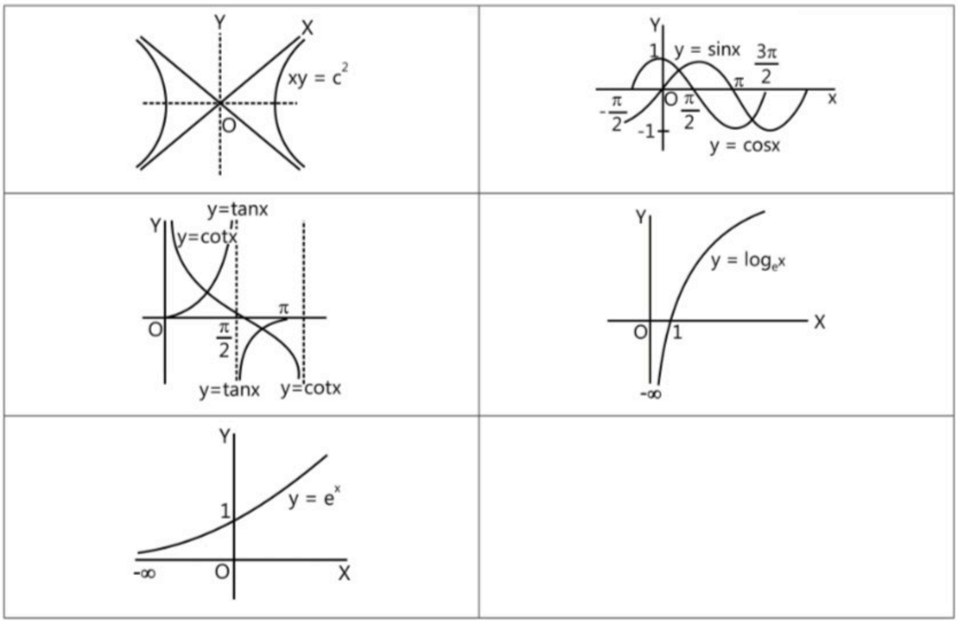
Figure 25.17

Illustration 15: Using integration, find the area of the region given below:

$\{(x, y): 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$ **(JEE ADVANCED)**

SKETCH OF STANDARD CURVES





4. STANDARD AREAS

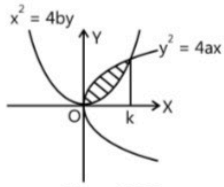


Figure 25.23

4.1 Area Bounded by Two Parabolas

Area between the parabolas $y^2 = 4ax$ and $x^2 = 4$; $a > 0$, $b > 0$, is

$$|A| = \frac{16ab}{3}$$

Illustration 20: Find the area bounded by $y = \sqrt{x}$ and $x = \sqrt{y}$. **(JEE MAIN)**

Sol: By using above mentioned formula.

Area bounded is shaded in the figure

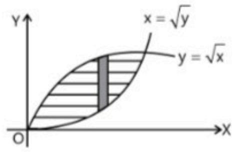


Figure 25.24

Here, $a = \frac{1}{4}$ and $b = \frac{1}{4}$

$\therefore$ Using the above formula, Area = (16 ab)/ 3

$$= \frac{16 \times (1/4) \times (1/4)}{3} = \frac{1}{3}$$

4.2 Area Bounded By Parabola and a Line

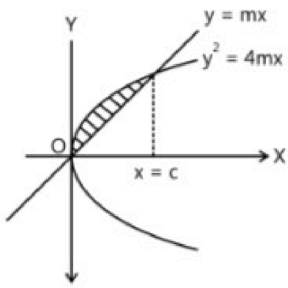


Figure 25.25

Area bounded by $y^2 = 4ax$ and $y = mx$; $a > 0$, $m > 0$ is $A = \frac{8a^2}{3m^3}$

Area bounded by $x^2 = 4ay$ and $y = mx$; $a > m > 0$

is $y = mx$; $a > m > 0$ $A = \frac{8a^2}{3m^3}$

Illustration 21: Find the area bounded by, $x^2 = y$ and $y = |x|$. **(JEE MAIN)**

Sol: Using above formula, i.e. $A = \frac{8a^2}{3m^3}$

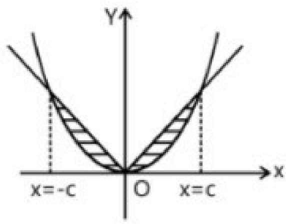


Figure 25.26

Area bounded is shaded in the Fig. 25.26.

Here, $a = 1/4$, $m = 1$

$\therefore$ Using the above formula, Area $= 2\left(\frac{8a^2}{3m^3}\right) = \frac{2 \times 8 \times \left(\frac{1}{4}\right)^2}{3 \times (1)^3} = \frac{1}{3}$

Illustration 22: Find the area bounded by $y^2 = x$ and $x = |y|$. **(JEE MAIN)**

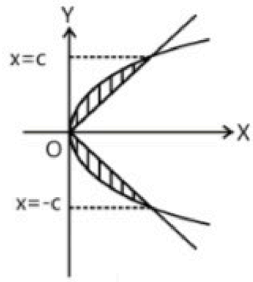


Figure 25.27

Sol: Here, $a = 1/4$, $m = 1$, and required area is divided in to two equal parts at above and below x – axis.

Hence required area will be $2\left(\frac{8a^2}{3m^3}\right)$.

$\therefore$ Using the above formula, Area $= 2\left(\frac{8a^2}{3m^3}\right) = \frac{2 \times 8 \times (1/4)^2}{3 \times (1)^3} = \frac{1}{3}$

4.3 Area Enclosed by Parabola and It's Chord

Area between $y^2 = 4ax$ and its double ordinate at $x = a$ is

Area of AOB $= \frac{2}{3}$ (area $\square$ABCD)

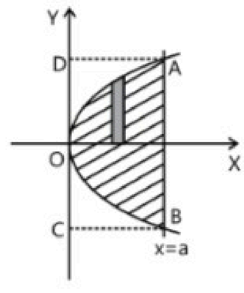


Figure 25.28

Illustration 23: Find the area bounded by $y = 2x - x^2$, $y + 3 = 0$. **(JEE MAIN)**

Sol: Here first obtain area of rectangle ABCD and after that by using above mentioned formula we will be get required area.

Solving $y = 2x - x^2$, $y + 3 = 0$, we get $x = -1$ or 3

Area (ABCD) $= 4 \times 4 = 16$.

$\therefore$ Required area $= \frac{2}{3} \times 16 = \frac{32}{3}$

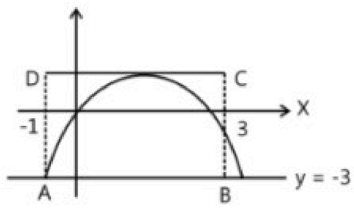


Figure 25.29

4.4 Area of an Ellipse

For an ellipse of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $A = \pi ab$

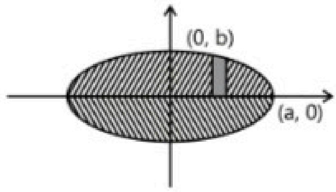


Figure 25.30

PLANCESS CONCEPTS

Try to remember some standard areas like for ellipse, parabola. These results are sometimes very helpful.

Vaibhav Gupta (JEE 2009 AIR 54)

5. SHIFTING OF ORIGIN

Area remains unchanged even if the coordinate axes are shifted or rotated or both. Hence shifting of origin / rotation of axes in many cases proves to be very convenient in finding the area.
For example: If we have a circle whose centre is not origin, we can find its area easily by shifting circle's centre.

Illustration 24: The line $3x + 2y = 13$ divides the area enclosed by the curve $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ into two parts. Find the ratio of the larger area to the smaller area. **(JEE ADVANCED)**

Sol: Given $9x^2+ 4y^2 - 18x - 16y - 11 = 0$ ... (i)

and, $3x + 2y = 13$ ... (ii)

$9(x^2 - 2x) + 4 (y^2 - 4y) = 11;$

$\Rightarrow 9[(x-1)^2 - 1] + 4[(y-2)^2 - 4] = 11$

$\Rightarrow 9(x-1)^2 + 4(y-2)^2 = 36$

$\Rightarrow \frac{(x-1)^2}{4}+\frac{(y-2)^2}{9}=1 \Rightarrow \frac{X^2}{4}+\frac{Y^2}{9}=1$ (where $X = x - 1$ and $Y = y - 2$)

Hence $3x + 2y = 13$

$\Rightarrow 3(X + 1) + 2(Y + 2) = 13$

$\Rightarrow 3X + 2Y = 6$

$\Rightarrow \frac{X}{2}+\frac{Y}{3}=1$

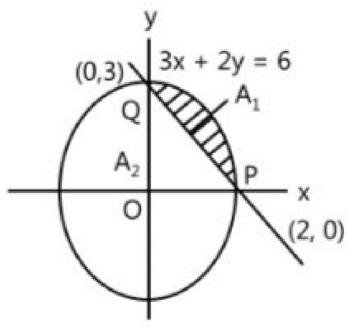


Figure 25.31

$\therefore$ Area of triangle OPQ = $1/2 \times 2 \times 3 = 3$

Also area of ellipse = π (semi major axes) (semi minor axis) = $\pi.3.2 = 6\pi$

$A_1 = \frac{6\pi}{4} - \text{area of } \Delta OPQ = \frac{3\pi}{2} - 3$

$A_2 = 3\left(\frac{6\pi}{4}\right) + \text{area of } \Delta OPQ = \frac{9\pi}{2} + 3$

Hence, $\frac{A_2}{A_1} = \frac{\frac{9\pi}{2}+3}{\frac{3\pi}{2}-3} = \frac{3\pi+2}{\pi-2}$

6. DETERMINATION OF PARAMETERS

In this type of questions, you will be given area of the curve bounded between some axes or points, and some parameter(s) will be unknown either in equation of curve or a point or an axis. You have to find the value of the parameter by using the methods of evaluating area.

Illustration 25: Find the value of c for which the area of the figure bounded by the curves $y = \frac{4}{x^2}$; $x = 1$ and $y = c$ is equal to $\frac{9}{4}$. **(JEE MAIN)**

Sol: By using method of evaluating area we can find out the value of c.

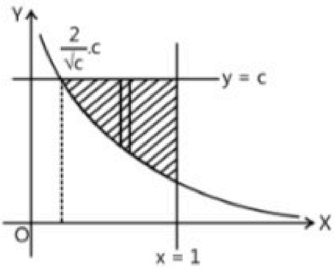


Figure 25.32

$$A = \int_{2/\sqrt{c}}^{1}\left(c - \frac{4}{x^2}\right)dx = \frac{9}{4};\ \left(cx + \frac{4}{x}\right)_{\frac{2}{\sqrt{c}}}^{1} = \frac{9}{4}$$

$$(c+4) - (2\sqrt{c} + 2\sqrt{c}) = \frac{9}{4};\ c - 4\sqrt{c} + 4 = \frac{9}{4}$$

$$\Rightarrow (\sqrt{c} - 2)^2 = \frac{9}{4} \Rightarrow (\sqrt{c} - 2) = \frac{3}{2} \text{ or } -\frac{3}{2}$$

Hence c = (49/4) or (1/4)

Illustration 26: Consider the two curves:

$C_1 : y = 1 + \cos x$, and $C_2 : y = 1 + \cos(x - \alpha)$ for $\alpha \in (0, \pi/2)$ and $x \in [0, p]$.

Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 and $x = 0$ is same as that of the area bounded by C_2, $y = 1$ and $x = \pi$. For this value of α, find the ratio in which the line $y = 1$ divides the area of the figure by the curves C_1, C_2 and $x = \pi$. **(JEE ADVANCED)**

Sol: Solve C_1 and C_2 to obtain the value of x, after that by following given condition we will be obtain required value of α.

Solving C_1 and C_2, we get

$$1 + \cos x = 1 + \cos(x - \alpha) \Rightarrow x = \alpha - x \quad \Rightarrow x = \frac{\alpha}{2}$$

According to the question,

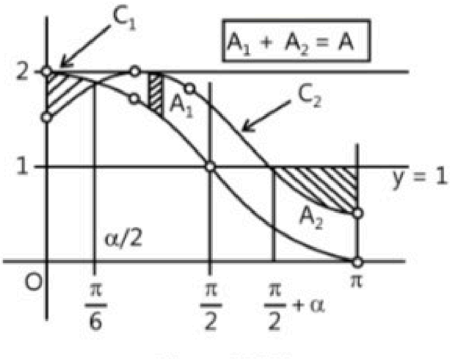


Figure 25.33

$$\int_0^{\alpha/2}(\cos x - \cos(x-\alpha))dx = -\int_{\frac{\pi}{2}+\alpha}^{\pi}(\cos(x-\alpha))dx$$

$$\Rightarrow \left[\sin x - \sin(x-\alpha)\right]_0^{\alpha/2} = \left[\sin(x-\alpha)\right]_{\pi}^{\frac{\pi}{2}+\alpha}$$

$$\Rightarrow \left[\sin\frac{\alpha}{2} - \sin\left(-\frac{\alpha}{2}\right)\right] - [0 - \sin(-\alpha)] = \sin\left(\frac{\pi}{2}\right) - \sin(\pi - \alpha)$$

$$\Rightarrow 2\sin\frac{\alpha}{2} - \sin\alpha = 1 - \sin\alpha \text{. Hence, } 2\sin\frac{\alpha}{2} = 1 \qquad \Rightarrow \alpha = \frac{\pi}{3}$$

7. AREA BOUNDED BY THE INVERSE FUNCTION

The area of the region bounded by the inverse of a given function can also be calculated using this method. The graph of inverse of a function is symmetric about the line $y = x$. We use this property to calculate the area. Hence, area of the function between $x = a$ to $x = b$, is equal to the area of inverse function from f(a) to f(b).

8. VARIABLE AREA

If y = f(x) is a monotonic function in (a, b), then the area of the function y = f(x) bounded by the lines at x = a, x = b, and the line y = f(c), [where $c \in (a, b)$] is minimum when $c = \frac{a+b}{2}$.

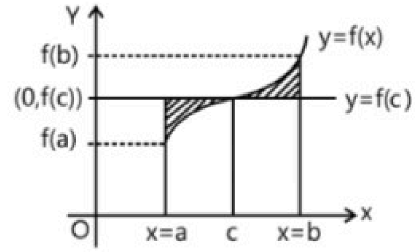


Figure 25.36

Proof: $A = \int_a^c f(c) - f(x)dx + \int_c^b (f(x) - f(c))dx$

$= f(c)\,(c-a) - \int_a^c (f(x))dx + \int_c^b (f(x))dx - f(c)(b-c)$

$= \{(c-a) - (b-c)\}\, f(c) + \int_c^b (f(x))dx - \int_a^c (f(x))dx$

$A = [2c - (a+b)]\ f(c) + \int_c^b (f(x))dx - \int_a^c (f(x))dx$

For maxima and minima $\frac{dA}{dc} = 0 \Rightarrow f'(c) = [2c - (a+b)] = 0$ (as $f'(c) = 0$) hence $c = \frac{a+b}{2}$ also $c < \frac{a+b}{2}, \frac{dA}{dc} < 0$ and $c > \frac{a+b}{2}, \frac{dA}{dc} > 0$ Hence A is minimum when $c = \frac{a+b}{2}$

9. AVERAGE VALUE OF A FUNCTION

In this section, we would study the average of a continuous function. This concept of average is frequently applied in physics and chemistry.

Average of a function f(x) between x = a to x = b is given by $y_{av} = \frac{1}{b-a}\int_a^b f(x)dx$

PLANCESS CONCEPTS

(a) Average value can be positive, negative or zero .

(b) If the function is defined in $(0, \infty)$, then $y_{av} = \lim_{b \to \infty} \frac{1}{b}\int_0^b f(x)dx$ provided the limit exists

(c) Root mean square value (RMS) is defined as $\rho = \left[\frac{1}{b-a}\int_a^b f^2(x)\,dx\right]^{\frac{1}{2}}$

(d) If a function is periodic then we need to calculate average of function in particular time period that is its overall mean.

Vaibhav Krishnan (JEE 2009 AIR 22)

Illustration 29: Find the average value of y^2 w.r.t. x for the curve $ay = b\sqrt{a^2 - x^2}$ between x = 0 & x = a. Also find the average value of y w.r.t. x^2 for $0 \le x \le a$. **(JEE MAIN)**

Sol: As average of a function f(x) between x = a to x = b is given by $y_{av} = \frac{1}{b-a}\int_a^b f(x)dx$

Let $f(x) = y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ Now $f(x)|_{av} = \frac{b^2}{a^2(a-0)}\int_0^a (a^2 - x^2)dx = \frac{2b^2}{3}$

Again y_{av} w.r.t. x^2 as $f(x)|_{av} = \frac{1}{(a^2-0)}\int_0^{a^2} y\, d(x^2) = \frac{b}{a^2 a}\int_0^{a^2}\sqrt{a^2 - x^2}\, dx^2 = \frac{b}{a^3}\int_0^{a^2} 2t^2\, dt = \frac{2ba^3}{3}$

10. DETERMINATION OF FUNCTION

Sometimes the area enclosed by a curve is given as a variable function and we have to find the function. The area function A_a^x satisfies the differential equation $\frac{dA_a^x}{dx} = f(x)$ with initial condition $A_a^a = 0$ i.e. derivative of the area function is the function itself. Thus we can easily find f(x) by differentiating area function.

PLANCESS CONCEPTS

If F(x) is integral of f(x) then, $A_a^x = \int f(x)dx = [F(x) + c]$

And since, $A_a^a = 0 = F(a) + c \quad \Rightarrow c = -F(a)$.

$\therefore A_a^x = F(x) - F(a)$. Finally by taking x = b we get, $A_a^b = F(b) - F(a)$

Note that this is true only if the function doesn't have any zeroes between a and b.

If the function has zero at c then area = $|F(b) - F(c)| + |F(c) - F(a)|$

Vaibhav Gupta (JEE 2009 AIR 54)

Illustration 30: The area from 0 to x under a certain graph is given to be $A = \sqrt{1+3x} - 1, x \ge 0$;

(a) Find the average rate of change of A w.r.t. x and x increases from 1 to 8.

(b) Find the instantaneous rate of change of A w.r.t. x at x = 5.

(c) Find the ordinate (height) y of the graph as a function of x.

(d) Find the average value of the ordinate (height) y, w.r.t. x as x increases from 1 to 8. **(JEE ADVANCED)**

Sol: Here by differentiating given area function we can obtain the main function.

(a) A(1) = 1, A(8) = 4 ; $\frac{A(8) - A(1)}{8-1} = \frac{3}{7}$

(b) $\left.\frac{dA}{dx}\right|_{x=5} = \left.\frac{1.\,3}{2\sqrt{1+3x}}\right|_{x=5} = \frac{3}{8}$

(c) $y = \frac{3}{2\sqrt{1+3x}}$

(d) $\frac{1}{(8-1)}\int_1^8 \frac{3}{2\sqrt{1+3x}}dx = \frac{1}{7}\int_1^8 \frac{3}{2\sqrt{1+3x}}dx = \frac{3}{7}$

Illustration 31: Let C_1 & C_2 be the graphs of the function $y = x^2$ & $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graphs of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 & C_3 at Q & R respectively (see figure). If for every position of P(on C_1), the area of the shaded regions OPQ & ORP are equal, determine the function f(x). **(JEE ADVANCED)**

Sol: Similar to the above mentioned method.

$$\int_0^{h^2}\left(\sqrt{y}-\frac{y}{2}\right)dy = \int_0^{h}(x^2 - f(x))\,dx$$ differentiate both sides w.r.t. h

$$\left(h-\frac{h^2}{2}\right)2h = h^2 - f(h)$$

$$f(h) = h^2 - \left(h-\frac{h^2}{2}\right)2h$$

$$= h^2 - h(2h - h^2) = h^2 - 2h^2 + h^3$$

$$f(h) = h^3 - h^2$$

$$f(x) = x^3 - x^2 = x^2(x-1)$$

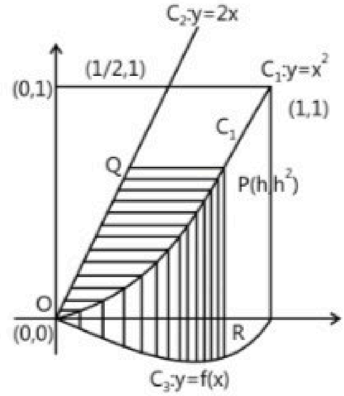


Figure 25.37

11. AREA ENCLOSED BY A CURVE EXPRESSED IN POLAR FORM

$r = a(1 + \cos\theta)$ (Cardioid)

$$A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{a^2}{2}\int_0^{2\pi} 4\cos^4\frac{\theta}{2}d\theta$$

Substitute $\frac{\theta}{2} = t$, $d\theta = 2dt$

$$A = a^2\int_0^{\pi} 4\cos^4 t\,dt = 8 \times \frac{3\pi a^2}{16}$$

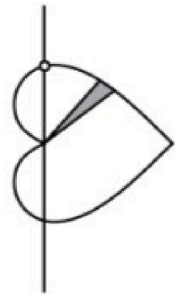


Figure 25.38

Illustration 32: Find the area enclosed by the curves $x = a\sin^3 t$ and $y = a\cos^3 t$. **(JEE MAIN)**

Sol:

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ and $dx = 3a\sin^2 t.\cos t.dt$

$$A = 4\int_0^{a} y\,dx \;;\; A = 4a^2\int_0^{\pi/2} 3\cos^3 t\sin^2 t\cos t\,dt$$

$$A = 12a^2\int_0^{\pi/2}\sin^2 t\cos^4 t\,dt = (12a^2)\,.\,\frac{1.3.1}{6.4.2}\,.\,\frac{\pi}{2} = \frac{12a^2\pi}{32} = \frac{3\pi a^2}{8}$$

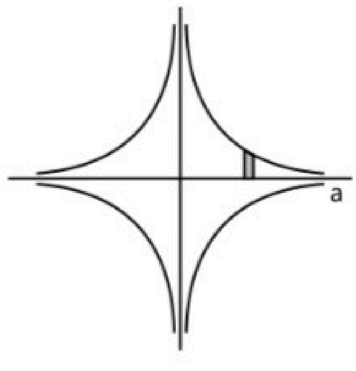


Figure 25.39