Exercises

Single Correct Answer Type

1. Let $f(x)=\lim_{n\to\infty}\frac{1}{n}\left(\left(x+\frac{1}{n}\right)^2+\left(x+\frac{2}{n}\right)^2+\ldots+\left(x+\frac{n-1}{n}\right)^2\right)$.
Then the minimum value of $f(x)$ is
(1) 1/4 (2) 1/6 (3) 1/9 (4) 1/12

2. If $S_n=\left[\frac{1}{1+\sqrt{n}}+\frac{1}{2+\sqrt{2n}}+\cdots+\frac{1}{n+\sqrt{n^2}}\right]$, then $\lim_{n\to\infty}S_n$ is equal to
(1) log 2 (2) log 4
(3) log 8 (4) none of these

3. The value of $\lim_{n\to\infty}\sum_{r=1}^{4n}\frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}$ is equal to
(1) $\frac{1}{35}$ (2) $\frac{1}{14}$ (3) $\frac{1}{10}$ (4) $\frac{1}{5}$

4. The value of
$$\lim_{n\to\infty}\frac{(1^2+2^2+\ldots+n^2)(1^3+2^3+\ldots+n^3)(1^4+2^4+\ldots+n^4)}{(1^5+2^5+\ldots+n^5)^2}$$
is equal to
(1) $\frac{3}{5}$ (2) $\frac{4}{5}$ (3) $\frac{2}{5}$ (4) $\frac{1}{5}$

5. The value of $\lim_{n\to\infty}\left[\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\cdots\tan\frac{n\pi}{2n}\right]^{1/n}$ is
(1) e (2) e^2 (3) 1 (4) e^3

6. $\int_{2-a}^{2+a}f(x)dx$ is equal to [where $f(2-\alpha)=f(2+\alpha)\ \forall\ \alpha\in R$]
(1) $2\int_2^{2+a}f(x)dx$ (2) $2\int_0^a f(x)dx$
(3) $2\int_2^2 f(x)dx$ (4) none of these

7. Let $f(x)=\min(\{x\},\{-x\})\ \forall\ x\in R$, where $\{\cdot\}$ denotes the fractional part of x. Then $\int_{-100}^{100}f(x)dx$ is equal to
(1) 50 (2) 100
(3) 200 (4) none of these

8. Which of the following is incorrect?
(1) $\int_{a+c}^{b+c}f(x)dx=\int_a^b f(x+c)dx$
(2) $\int_{ac}^{bc}f(x)dx=c\int_a^b f(cx)dx$
(3) $\int_{-a}^{a}f(x)dx=\frac{1}{2}\int_{-a}^{a}(f(x)+f(-x)\,dx$
(4) None of these

9. $\int_{-1}^{1/2}\frac{e^x\left(2-x^2\right)dx}{(1-x)\sqrt{1-x^2}}$ is equal to
(1) $\frac{\sqrt{e}}{2}(\sqrt{3}+1)$ (2) $\frac{\sqrt{3e}}{2}$
(3) $\sqrt{3e}$ (4) $\sqrt{\frac{e}{3}}$

10. If $\int_{\log 2}^{x}\frac{dx}{\sqrt{e^x-1}}=\frac{\pi}{6}$, then x is equal to
(1) 4 (2) ln 8
(3) ln 4 (4) none of these

11. $\int_{5/2}^{5}\frac{\sqrt{(25-x^2)^3}}{x^4}dx$ is equal to
(1) $\frac{\pi}{6}$ (2) $\frac{2\pi}{3}$ (3) $\frac{5\pi}{6}$ (4) $\frac{\pi}{3}$

12. If $f(x)$ satisfies the condition of Rolle's theorem in [1, 2], then $\int_1^2 f'(x)\,dx$ is equal to
(1) 1 (2) 3
(3) 0 (4) none of these

13. The value of the integral $\int_0^{\log 5}\frac{e^x\sqrt{e^x-1}}{e^x+3}dx$ is
(1) $3+2\pi$ (2) $4-\pi$
(3) $2+\pi$ (4) none of these

14. The value of the integral $\int_0^1\frac{dx}{x^2+2x\cos\alpha+1}$, $0<\alpha<\pi$ is equal to
(1) $\sin\alpha$ (2) $\alpha\sin\alpha$
(3) $\frac{\alpha}{2\sin\alpha}$ (4) $\frac{\alpha}{2}\sin\alpha$

15. $\int_0^{\infty}\frac{dx}{[x+\sqrt{x^2+1}]^3}$ is equal to
(1) $\frac{3}{8}$ (2) $\frac{1}{8}$
(3) $-\frac{3}{8}$ (4) none of these

16. If $f(y)=e^y$, $g(y)=y$, $y>0$, and $F(t)=\int_0^t f(t-y)\,g(y)\,dy$ then
(1) $F(t)=e^t-(1+t)$ (2) $F(t)=te^t$
(3) $F(t)=te^{-t}$ (4) $F(t)=1-e^t(1+t)$

17. If $P(x)$ is a polynomial of the least degree that has maximum equal to 6 at $x=1$, and a minimum equal to 2 at $x=3$, then $\int_0^1 P(x)dx$ equals
(1) $\frac{17}{4}$ (2) $\frac{13}{4}$ (3) $\frac{19}{4}$ (4) $\frac{5}{4}$

Exercises

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1. (4) $f(x) = \lim_{n \to \infty} \frac{1}{n}\left(\left(x + \frac{1}{n}\right)^2 + \left(x + \frac{2}{n}\right)^2 + \ldots + \left(x + \frac{n-1}{n}\right)^2\right)$

$$= \int_0^1 (x + y)^2\, dy$$

$$= x^2 + x + \frac{1}{3}$$

$$\therefore \quad f(x) = \left(x + \frac{1}{2}\right)^2 + \frac{1}{3} - \frac{1}{4}$$

$$\therefore \quad f(x)_{\min} = \frac{1}{12}$$

2. (2) $\lim_{n \to \infty} S_n = \lim_{n \to \infty}\left[\frac{1}{1 + \sqrt{n}} + \frac{1}{2 + \sqrt{2n}} + \cdots + \frac{1}{n + \sqrt{n^2}}\right]$

$$= \lim_{n \to \infty} \frac{1}{n}\left[\frac{1}{\frac{1}{n} + \frac{1}{\sqrt{n}}} + \frac{1}{\frac{2}{n} + \sqrt{\frac{2}{n}}} + \cdots + \frac{1}{\frac{n}{n} + \sqrt{\frac{n}{n}}}\right]$$

$$= \int_0^1 \frac{dx}{\sqrt{x}\left(\sqrt{x} + 1\right)}$$

Put $\sqrt{x} = z$ or $\frac{1}{2\sqrt{x}}\, dx = dz$

$$\therefore \quad \lim_{n \to \infty} S_n = \int_0^1 \frac{2dz}{z + 1} = 2\,[\log(z + 1)]_0^1$$

$$= 2(\log 2 - \log 1)$$

$$= 2 \log 2 = \log 4$$

3. (3) $\lim_{n\to\infty}\sum_{r=1}^{4n}\frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}$

$$T_r=\frac{1}{\sqrt{\frac{r}{n}}\,n\left(3\sqrt{\frac{r}{n}}+4\right)^2}$$

$$\therefore\quad S=\lim_{n\to\infty}\frac{1}{n}\sum_{1}^{4n}\frac{1}{\left(3\sqrt{\frac{r}{n}}+4\right)^2\sqrt{\frac{r}{n}}}=\int_0^4\frac{dx}{\sqrt{x}(3\sqrt{x}+4)^2}$$

Put $3\sqrt{x}+4=t$

or $\frac{3}{2}\frac{1}{\sqrt{x}}dx=dt$

$$\therefore\quad S=\frac{2}{3}\int_4^{10}\frac{dt}{t^2}=\frac{2}{3}\left[-\frac{1}{t}\right]_{10}^{4}=\frac{1}{10}$$

4. (1) $\lim_{n\to\infty}\dfrac{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^2}{n^2}\right)\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^3}{n^3}\right)\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^4}{n^4}\right)}{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^5}{n^5}\right)^2}$

$$=\frac{\int_0^1 x^2dx\cdot\int_0^1 x^3dx\cdot\int_0^1 x^4dx}{\left(\int_0^1 x^5dx\right)^2}$$

$$=\frac{\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}}{\frac{1}{6}\times\frac{1}{6}}=\frac{36}{60}=\frac{3}{5}$$

5. (3) Let $A=\lim_{n\to\infty}\left[\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\cdots\tan\frac{n\pi}{2n}\right]^{1/n}$

$$\therefore\quad \log A=\lim_{n\to\infty}\frac{1}{n}\left[\log\tan\frac{\pi}{2n}+\log\tan\frac{2\pi}{2n}+\dots+\log\tan\frac{n\pi}{2n}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\tan\frac{\pi r}{2n}$$

$$=\int_0^1\log\tan\left(\frac{\pi}{2}x\right)dx$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\log\tan y\,dy \qquad (1)$$

$$\left[\text{Putting }\frac{1}{2}\pi x=y\text{ and }dx=(2/\pi)\,dy\right]$$

Now, let $I=\int_0^{\pi/2}\log\tan y\,dy$

$$\therefore\quad I=\int_0^{\pi/2}\log\tan\left(\frac{1}{2}\pi-y\right)dy$$

$$=\int_0^{\pi/2}\log\cot y\,dy$$

$$=-\int_0^{\pi/2}\log\tan y\,dy=-I$$

or $I+I=0$ or $2I=0$ or $I=0$

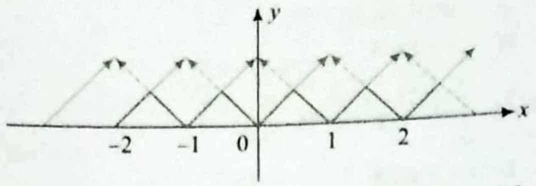
Thus, from equation (1), $\log A=0$ or $A=e^0=1$.

6. (1) $f(2-\alpha)=f(2+\alpha)$

Thus, function is symmetric about the line $x=2$.

$$\int_{2-a}^{2+a}f(x)dx=2\int_{2}^{2+a}f(x)dx$$

7. (1)



The graph with solid line is the graph of $f(x)=\{x\}$ and the graph with dotted lines is the graph of $f(x)=\{-x\}$. Now, the graph of $\min(\{x\},\{-x\})$ is the graph with dark solid lines.

$\int_{-100}^{100}f(x)dx=$ Area of 200 triangles shown as solid dark lines in the diagram

$$=200\frac{1}{2}(1)\left(\frac{1}{2}\right)=50$$

8. (4) $I_1=\int_{a+c}^{b+c}f(x)dx$. Putting $x=t+c$ and $dx=dt$, we get

$$I_1=\int_a^b f(t+c)dt=\int_a^b f(x+c)dx$$

Now, $I_2=\int_{ac}^{bc}f(x)dx$

Putting $x=tc$ and $dx=c\,dt$, we get

$$I_2=c\int_a^b f(ct)dt=c\int_a^b f(cx)dx$$

$$f(x)=\frac{1}{2}(f(x)+f(-x)+f(x)-f(-x))$$

$$\therefore\quad I_3=\int_{-a}^{a}f(x)dx$$

$$=\frac{1}{2}\int_{-a}^{a}(f(x)+f(-x)+f(x)-f(-x))dx$$

$$=\frac{1}{2}\int_{-a}^{a}(f(x)+f(-x))dx+\frac{1}{2}\int_{-a}^{a}(f(x)-f(-x))dx$$

$$=\frac{1}{2}\int_{-a}^{a}(f(x)+f(-x))dx$$

as $f(x)+f(-x)$ is even and $f(x)-f(-x)$ is odd.

9. (3) $\int_{-1}^{1/2}\frac{e^x(2-x^2)dx}{(1-x)\sqrt{1-x^2}}=\int_{-1}^{1/2}\frac{e^x(1-x^2+1)}{(1-x)\sqrt{1-x^2}}$

$$=\int_{-1}^{1/2}e^x\left[\sqrt{\frac{1+x}{1-x}}+\frac{1}{(1-x)\sqrt{1-x^2}}\right]dx$$

$$=e^x\sqrt{\frac{1+x}{1-x}}\Bigg|_{-1}^{1/2}=\sqrt{3e}$$

10. (3) $\frac{\pi}{6}=\int_{\log 2}^{x}\frac{dx}{\sqrt{e^x-1}}=\int_{\log 2}^{x}\frac{e^{x/2}dx}{e^{x/2}\sqrt{(e^{x/2})^2-1}}$

$$=2\left[\sec^{-1}e^{x/2}\right]_{\log 2}^{x}$$

$$=2\left[\sec^{-1}e^{x/2}-\sec^{-1}\sqrt{2}\right]$$

or $\frac{\pi}{12}+\frac{\pi}{4}=\sec^{-1}e^{x/2}$

or $\frac{\pi}{3}=\sec^{-1}e^{x/2}$

or $e^{x/2}=2$

or $x/2=\log 2$

or $x=\log 4$

11. (4) $I=\int_{5/2}^{5}\frac{\sqrt{(25-x^2)^3}}{x^4}dx$

Let $x=5\sin\theta$

$\therefore\quad dx=5\cos\theta\,d\theta$

$\therefore\quad I=\int_{\pi/6}^{\pi/2}\frac{\sqrt{(25-25\sin^2\theta)^3}}{5^4\sin^4\theta}.5\cos\theta\,d\theta$

$=\int_{\pi/6}^{\pi/2}\frac{5^3\cos^3\theta.5\cos\theta}{5^4\sin^4\theta}d\theta$

$=\int_{\pi/6}^{\pi/2}\cot^2\theta(\operatorname{cosec}^2\theta-1)d\theta$

$=\int_{\pi/6}^{\pi/2}\cot^2\theta\operatorname{cosec}^2\theta\,d\theta-\int_{\pi/6}^{\pi/2}\cot^2\theta\,d\theta$

$=\int_{\pi/6}^{\pi/2}\cot^2\theta\operatorname{cosec}^2\theta\,d\theta-\int_{\pi/6}^{\pi/2}(\operatorname{cosec}^2\theta-1)d\theta$

$=\left[-\frac{\cot^3\theta}{3}+\cot\theta+\theta\right]_{\pi/6}^{\pi/2}$

$=-0+0+\frac{\pi}{2}-\left(-\frac{3\sqrt{3}}{3}+\sqrt{3}+\frac{\pi}{6}\right)=\frac{\pi}{3}$

2. (3) As $f(x)$ satisfies the conditions of Rolle's theorem in [1, 2], $f(x)$ is continuous in the interval and $f(1)=f(2)$.

Therefore, $\int_1^2 f'(x)dx=[f(x)]_1^2=f(2)-f(1)=0$

. (2) Putting $e^x-1=t^2$ in the given integral, we have

$\int_0^{\log 5}\frac{e^x\sqrt{e^x-1}}{e^x+3}dx=2\int_0^2\frac{t^2}{t^2+4}dt=2\left(\int_0^2 1dt-4\int_0^2\frac{dt}{t^2+4}\right)$

$=2\left[\left(t-2\tan^{-1}\left(\frac{t}{2}\right)\right)_0^2\right]$

$=2[(2-2\times\pi/4)]=4-\pi$

(3) Given integral

$=\int_0^1\frac{dx}{(x+\cos\alpha)^2+(1-\cos^2\alpha)}$

$=\int_0^1\frac{dx}{(x+\cos\alpha)^2+\sin^2\alpha}$

$=\frac{1}{\sin\alpha}\left|\tan^{-1}\frac{x+\cos\alpha}{\sin\alpha}\right|_0^1$

$=\frac{1}{\sin\alpha}\left[\tan^{-1}\frac{1+\cos\alpha}{\sin\alpha}-\tan^{-1}\frac{\cos\alpha}{\sin\alpha}\right]$

$=\frac{1}{\sin\alpha}\left[\tan^{-1}\cot\frac{\alpha}{2}-\tan^{-1}(\cot\alpha)\right]$

$=\frac{1}{\sin\alpha}\left[\tan^{-1}\tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)-\tan^{-1}\tan\left(\frac{\pi}{2}-\alpha\right)\right]$

$=\frac{1}{\sin\alpha}\left[\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)-\left(\frac{\pi}{2}-\alpha\right)\right]$

$=\frac{\alpha}{2\sin\alpha}$

15. (1) Putting $x=\tan\theta$, we get

$\int_0^{\pi/2}\frac{dx}{[x+\sqrt{x^2+1}]^3}=\int_0^{\pi/2}\frac{\sec^2\theta d\theta}{(\tan\theta+\sec\theta)^3}$

$=\int_0^{\pi/2}\frac{\cos\theta}{(1+\sin\theta)^3}d\theta$

$=\left[-\frac{1}{2(1+\sin\theta)^2}\right]_0^{\pi/2}=-\frac{1}{8}+\frac{1}{2}=\frac{3}{8}$

16. (1) We have $f(y)=e^y$, $g(y)=y$, $y>0$

$F(t)=\int_0^t f(t-y)g(y)dy$

$=\int_0^t e^{t-y}y\,dy$

$=e^t\int_0^t e^{-y}y\,dy$

$=e^t\left(\left[-ye^{-y}\right]_0^t+\int_0^t e^{-y}\,dy\right)$

$=e^t\left(-te^{-t}-\left[e^{-y}\right]_0^t\right)$

$=e^t\left(-te^{-t}-e^{-t}+1\right)$

$=e^t-(1+t)$

17. (3) The polynomial function is differentiable everywhere. Therefore, the points of extremum can only be the roots of the derivative. Further, the derivative of a polynomial is a polynomial. The polynomial of the least degree with roots $x=1$ and $x=3$ has the form $a(x-1)(x-3)$.

Hence, $P'(x)=a(x-1)(x-3)$.

Since at $x=1$, we must have $P(1)=6$.

$P(x)=\int_1^x P'(x)dx+6$

$=a\int_1^x(x^2-4x+3)dx+6$

$=a\left(\frac{x^3}{3}-2x^2+3x-\frac{4}{3}\right)+6$

Also, $P(3)=2$. So, $a=3$. Hence, $P(x)=x^3-6x^2+9x+2$.

Thus, $\int_0^1 P(x)dx=\frac{1}{4}-2+\frac{9}{2}+2=\frac{19}{4}$

18. (4) Since $a^2I_1-2aI_2+I_3=0$,

$\int_0^1(a-x)^2f(x)dx=0$

Hence, there is no such positive function $f(x)$.