COMPLEX NUMBER

1. DEFINITION :

Complex numbers are defined as expressions of the form $a + ib$ where $a, b \in R$ & $i = \sqrt{-1}$. It is denoted by z i.e. $z = a + ib$. 'a' is called real part of z (Re z) and 'b' is called imaginary part of z (Im z).

Every Complex Number Can Be Regarded As

Purely real	Purely imaginary	Imaginary
if $b = 0$	if $a = 0$	if $b \neq 0$

Note :

(i) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complex Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(ii) Zero is both purely real as well as purely imaginary but not imaginary.

(iii) $i = \sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(iv) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non-negative.

2. CONJUGATE COMPLEX :

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

Note that :

(i) $z + \bar{z} = 2\,\text{Re}(z)$

(ii) $z - \bar{z} = 2i\,\text{Im}(z)$

(iii) $z\bar{z} = a^2 + b^2$ which is real

(iv) If z is purely real then $z - \bar{z} = 0$

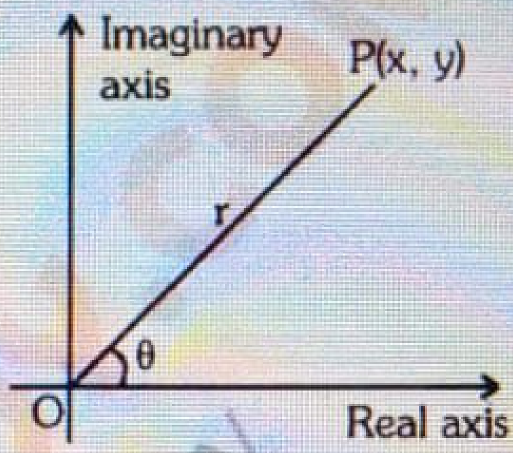
(v) If z is purely imaginary then $z + \bar{z} = 0$

3. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

(a) Cartesian Form (Geometrical Representation) :

Every complex number $z = x + iy$ can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).

Length OP is called **modulus** of the complex number denoted by $|z|$ & θ is called the **argument or amplitude**.



e.g. $|z| = \sqrt{x^2 + y^2}$ & $\theta = \tan^{-1}\dfrac{y}{x}$ (angle made by OP with positive x–axis)

Geometrically $|z|$ represents the distance of point P from origin. $(|z| \geq 0)$

(b) Trigonometric / Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as $\text{CiS}\,\theta$.

Euler's formula :

The formula $e^{ix} = \cos x + i\sin x$ is called Euler's formula.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ are known as Euler's identities.

(c) Exponential Representation :

Let z be a complex number such that $|z| = r$ & $\arg z = \theta$, then $z = r.e^{i\theta}$

4. IMPORTANT PROPERTIES OF CONJUGATE :

(a) $\overline{(\bar{z})} = z$

(b) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(c) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(d) $\overline{z_1\, z_2} = \bar{z}_1 \cdot \bar{z}_2$

(e) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$; $z_2 \neq 0$

(f) If $f(\alpha + i\beta) = x + iy \Rightarrow f(\alpha - i\beta) = x - iy$

5. IMPORTANT PROPERTIES OF MODULUS :

(a) $|z| \geq 0$ **(b)** $|z| \geq \text{Re}(z)$ **(c)** $|z| \geq \text{Im}(z)$

(d) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$ **(e)** $z\bar{z} = |z|^2$ **(f)** $|z_1 z_2| = |z_1| . |z_2|$

(g) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$ **(h)** $|z^n| = |z|^n$

(i) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$
or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

(j) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

(k) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ **[Triangle Inequality]**

(l) $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$ **[Triangle Inequality]**

(m) If $\left|z + \frac{1}{z}\right| = a$ $(a > 0)$, then $\max|z| = \frac{a + \sqrt{a^2 + 4}}{2}$

& $\min|z| = \frac{1}{2}\left(\sqrt{a^2 + 4} - a\right)$

6. IMPORTANT PROPERTIES OF AMPLITUDE :

(a) $\text{amp}(z_1 . z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi$; $k \in I$

(b) $\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi$; $k \in I$

(c) $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$, where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(d) $\log(z) = \log(re^{i\theta}) = \log r + i\theta = \log|z| + i\,\text{amp}(z)$

7. DE'MOIVER'S THEOREM :

The value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ if 'n' is integer & it is one of the values of $(\cos\theta + i\sin\theta)^n$ if n is a rational number of the form p/q, where p & q are co-prime.

Note : Continued product of roots of a complex quantity should be determined using theory of equation.

8. CUBE ROOT OF UNITY :

(a) The cube roots of unity are 1 , $\omega = \dfrac{-1+i\sqrt{3}}{2} = e^{i2\pi/3}$

& $\omega^2 = \dfrac{-1-i\sqrt{3}}{2} = e^{i4\pi/3}$

(b) $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$, in general

$$1+\omega^r+\omega^{2r} = \begin{cases} 0 & \text{r is not integral multiple of 3} \\ 3 & \text{r is multiple of 3} \end{cases}$$

(c) $a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$a^3 + b^3 = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$

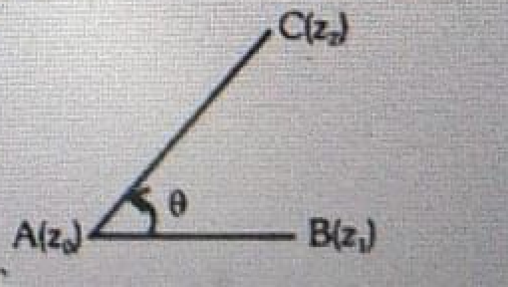
$x^2 + x + 1 = (x - \omega)(x - \omega^2)$

9. SQUARE ROOT OF COMPLEX NUMBER :

$$\sqrt{a+ib} = \pm\left\{\frac{\sqrt{|z|+a}}{2} + i\frac{\sqrt{|z|-a}}{2}\right\} \text{ for } b > 0$$

& $\pm\left\{\dfrac{\sqrt{|z|+a}}{2} - i\dfrac{\sqrt{|z|-a}}{2}\right\}$ for $b < 0$ where $|z| = \sqrt{a^2+b^2}$.

10. ROTATION :

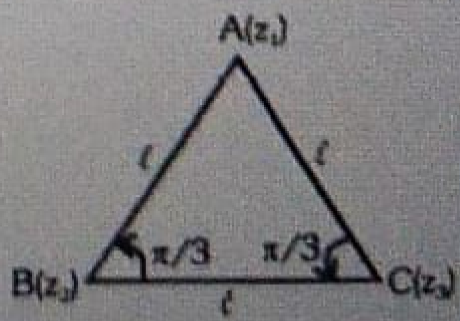


$$\frac{z_2 - z_0}{|z_2 - z_0|} = \frac{z_1 - z_0}{|z_1 - z_0|}e^{i\theta}$$

Take θ in anticlockwise direction

11. RESULT RELATED WITH TRIANGLE :

(a) Equilateral triangle :

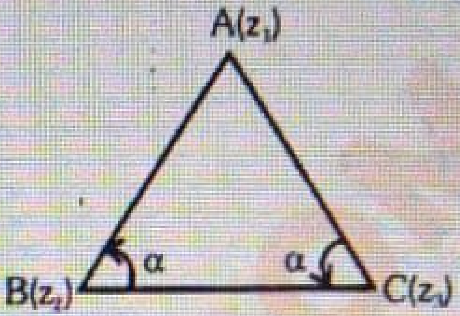


$$\frac{z_1 - z_2}{\ell} = \frac{z_3 - z_2}{\ell}e^{i\pi/3} \quad \text{......(i)}$$

Also $\dfrac{z_2 - z_3}{\ell} = \dfrac{z_1 - z_3}{\ell}.e^{i\pi/3}$ ......(ii) from (i) & (ii)

$$\Rightarrow \; z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

or $$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$



(b) Isosceles triangle :

$$4\cos^2\alpha\,(z_1 - z_2)(z_3 - z_1) = (z_3 - z_2)^2$$

(c) Area of triangle ΔABC given by modulus of $\frac{1}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$

12. EQUATION OF LINE THROUGH POINTS z_1 & z_2 :

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_2 - z_1) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$$

$$\Rightarrow z(\bar{z}_1 - \bar{z}_2)i + \bar{z}(z_2 - z_1)i + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

Let $(z_2 - z_1)i = a$, then equation of line is $\boxed{\bar{a}z + a\bar{z} + b = 0}$ where $a \in$

C & $b \in R$.

Note :

(i) Complex slope of line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{a}{\bar{a}}$

(ii) Two lines with slope μ_1 & μ_2 are parallel or perpendicular if $\mu_1 = \mu_2$ or $\mu_1 + \mu_2 = 0$

(iii) Length of perpendicular from point $A(\alpha)$ to line $\bar{a}z + a\bar{z} + b = 0$

is $\frac{|\bar{a}\alpha + a\bar{\alpha} + b|}{2|a|}$.

13. EQUATION OF CIRCLE :

(a) Circle whose centre is z_0 & radii = r

$$|z - z_0| = r$$

(b) General equation of circle

$$z\bar{z} + a\bar{z} + \bar{a}z + b = 0$$

centre '$-a$' & radii $= \sqrt{|a|^2 - b}$

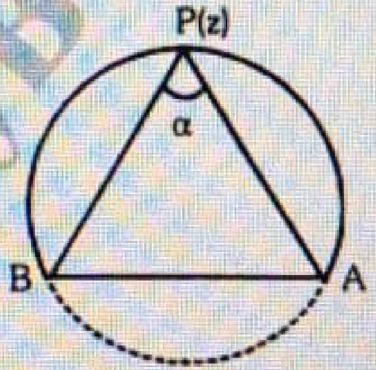
(c) Diameter form $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

or $\quad \arg\left(\frac{z - z_1}{z - z_2}\right) = \pm\frac{\pi}{2}$

(d) Equation $\left|\frac{z - z_1}{z - z_2}\right| = k$ represent a circle if $k \neq 1$ and a straight line if $k = 1$.

(e) Equation $|z - z_1|^2 + |z - z_2|^2 = k$

represent circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$



(f) $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha \quad 0 < \alpha < \pi, \ \alpha \neq \frac{\pi}{2}$

represent a segment of circle passing through $A(z_1)$ & $B(z_2)$

14. STANDARD LOCI :

(a) $|z - z_1| + |z - z_2| = 2k$ (a constant) represent

(i) if $2k > |z_1 - z_2|$ $\Rightarrow$ An ellipse

(ii) If $2k = |z_1 - z_2|$ $\Rightarrow$ A line segment

(iii) If $2k < |z_1 - z_2|$ $\Rightarrow$ No solution

(b) Equation $\big||z - z_1| - |z - z_2|\big| = 2k$ (a constant) represent

(i) If $2k < |z_1 - z_2|$ $\Rightarrow$ A hyperbola

(ii) If $2k = |z_1 - z_2|$ $\Rightarrow$ A line ray

(iii) $2k > |z_1 - z_2|$ $\Rightarrow$ No solution