

43. $\int \sqrt{\tan x} dx$ (Hint: Put $\tan x = t^2$)



$$\int \sqrt{\tan x} dx$$

Let $\tan x = t^2$

$$\sec^2 x dx = 2t dt$$

$$\Rightarrow dx = \frac{2t}{t^2+1} dt$$

So,

$$\int \frac{2t^2}{t^2+1} dt$$

$$= \int \frac{2 dt}{t^2 + \frac{1}{t^2}} = \int \frac{(1 - \frac{1}{t^2} + 1 + \frac{1}{t^2}) dt}{t^2 + \frac{1}{t^2}}$$

$$= \int \frac{(1 - \frac{1}{t^2}) dt}{(t + \frac{1}{t})^2 - 2} + \int \frac{(1 + \frac{1}{t^2}) dt}{(t - \frac{1}{t})^2 + 2}$$

$$\downarrow$$

$$t + \frac{1}{t} = u$$

$$(1 - \frac{1}{t^2}) dt = du$$

So,

$$\int \frac{du}{u^2 - 2}$$

$$\downarrow$$

$$t - \frac{1}{t} = k$$

$$(1 + \frac{1}{t^2}) dt = dk$$

So,

$$\int \frac{dk}{k^2 + 2}$$

$$= \frac{1}{2\sqrt{2}} \ln \left| \frac{u - \sqrt{2}}{u + \sqrt{2}} \right| + \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{k}{\sqrt{2}} \right)$$

So,

$$\frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{\tan x} + \frac{1}{\sqrt{\tan x}} - \sqrt{2}}{\sqrt{\tan x} + \frac{1}{\sqrt{\tan x}} + \sqrt{2}} \right| + \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{\tan x} - \frac{1}{\sqrt{\tan x}}}{\sqrt{2}} \right)$$