Find whether the following series is convergent or divergent:

$$x^2 + \frac{2^2}{3.4}x^4 + \frac{2^2 4^2}{3.4.5.6}x^6 + \frac{2^2 4^2 6^2}{3.4.5.6.7.8}x^8 + \ldots$$

Now put the value of $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{T_{n+1}}{T_n} = \lim_{n\to\infty} \frac{(2n+2)^2 x^2}{(2n+3)(2n+4)}$$

$$= \lim_{n\to\infty} \frac{\left(2+\frac{2}{n}\right)^2 x^2}{\left(2+\frac{3}{n}\right)\left(2+\frac{4}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(2+\frac{2}{\infty}\right)^2 x^2}{\left(2+\frac{3}{\infty}\right)\left(2+\frac{4}{\infty}\right)}$$

$$= \frac{(2+0)^2 x^2}{(2+0)(2+0)}$$

$$= x^2$$

Now for the value of $x > 1$, S_n diverges

For $-1 < x < 1$, S_n converges

And if $x = 1$ the ratio test fails

As already explained, if the ratio test fails, we use Raabe's Test

Raabe's test is given by the formula $\lim_{n\to\infty}\left[\frac{T_n}{T_{n+1}} - 1\right]n$; hence by applying in this series where $x = 1$ we get

$$\lim_{n\to\infty}\left[\frac{T_n}{T_{n+1}} - 1\right]n = \lim_{n\to\infty}\left[\frac{(2n+3)(2n+4)}{(2n+2)^2} - 1\right]n$$

$$= \lim_{n\to\infty}\left[\frac{4n^2 + 8n + 6n + 12}{4n^2 + 8n + 4} - 1\right]n$$

$$= \lim_{n\to\infty}\left[\frac{4n^2 + 14n + 12 - 4n^2 - 8n - 4}{4n^2 + 8n + 4}\right]n$$

$$= \lim_{n\to\infty} n\left[\frac{6n + 8}{4n^2 + 8n + 4}\right]$$

$$= \lim_{n\to\infty}\left[\frac{6n^2 + 8n}{4n^2 + 8n + 4}\right]$$

Now taking common from both the numerator and the denominator

$$\lim_{n\to\infty}\left[\frac{T_n}{T_{n+1}} - 1\right]n = \lim_{n\to\infty}\left[\frac{6n^2 + 8n}{4n^2 + 8n + 4}\right]$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2}\left[\frac{6+\frac{8}{n}}{4+\frac{8}{n}+\frac{4}{n^2}}\right]$$

Now put the value of$n \to \infty$, we get

$$\lim_{n\to\infty}\left[\frac{T_n}{T_{n+1}}-1\right]n = \lim_{n\to\infty} \frac{n^2}{n^2}\left[\frac{6+\frac{8}{n}}{4+\frac{8}{n}+\frac{4}{n^2}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{6+\frac{8}{\infty}}{4+\frac{8}{\infty}+\frac{4}{\infty^2}}\right]$$

$$= \left[\frac{6+0}{4+0+0}\right]$$

$$= \frac{6}{4}$$

$$= \frac{3}{2}$$

Since the value of $n = \frac{3}{2} > 1$

Hence the series is convergent at $x = 1$