PYQ4

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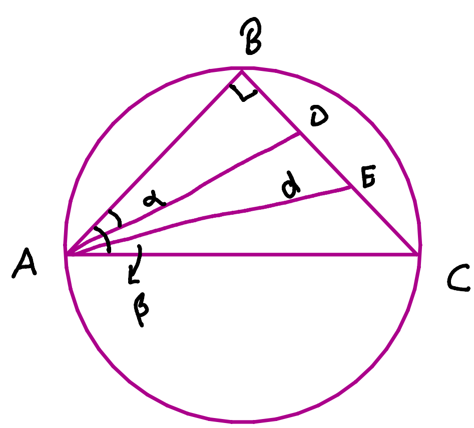
Question 4. A circle passes through 3 points A, B, C with line segment AC as diameter. A line passing through A intersects the chord BC at a point D inside the circle. If angles DAB and CAB are α (alpha) and β (beta) respectively and the distance between point A and the mid-point of line segment DC is d, prove that the area of the circle is:→

$$\frac{\pi d^2 \cos^2\alpha}{\{\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos(\beta-\alpha)\}}$$

[Subjective IIT-JEE 1996, 5]

Solution.



Let $AC = 2r$ and $AE = d$.

$\therefore AB = 2r\cos\beta$ and $AD = 2r.\frac{\cos\beta}{\cos\alpha}$

Now, $DE = \frac{(BC - BD)}{2}$

$\therefore BC = 2r\sin\beta$ —— ①

$\therefore BC = 2r\sin\beta$ —— (1)

and $\quad BD = AB\sin\alpha = 2r\cos\beta\tan\alpha$ —— (2)

$\therefore DE = \frac{1}{2}\left[2r\sin\beta - 2r\cos\beta\tan\alpha\right]$

$= \frac{r\sin(\beta-\alpha)}{\cos\alpha}$ —— (3)

Now, in ΔADC,

$$AC^2 + AD^2 = (AB^2 + BC^2) + AB^2 + BD^2$$

$$= 2AB^2 + BC^2 + BD^2$$

$$= 2AB^2 + (BD + 2DE)^2 + BD^2$$

$$= 2AB^2 + 2BD^2 + 4BD.DE + 4DE^2$$

$$= 2AB^2 + 2DE^2 + 2[BD^2 + 2BD.DE + DE^2]$$

$$= 2AB^2 + 2DE^2 + 2[BD + DE]^2$$

$$= (2AB^2 + 2BE)^2 + 2DE^2$$

$$\Rightarrow \quad AC^2 + AD^2 = 2AE^2 + 2DE^2$$

$$\therefore \quad 4r^2 + 4r^2.\frac{\cos^2\beta}{\cos^2\alpha} = 2d^2 + \frac{2r^2}{\cos^2\alpha}.\sin^2(\beta-\alpha)$$

$$\Rightarrow \frac{r^2}{\cos^2\alpha}\left[\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos(\beta-\alpha)\right] = d^2$$

$\therefore$ Area of circle $= \pi r^2$

$$= \frac{\pi d^2 \cos^2\alpha}{\cos^2\alpha + \cos^2\beta + 2\cos\alpha \cdot \cos\beta \cos(\beta - \alpha)}$$

Proved