Definite Integration of even / odd functions

* A function is even if it is symmetric about the vertical y-axis, i.e.,

$$f(-x) = f(x)$$

* A function is odd if it is symmetric about the origin, i.e.,

$$f(-x) = -f(x)$$

→ In order to use the special even or odd function rules for definite integrals, the interval must be of the form $[-a, a]$.

Rules:

* When $f(x)$ is even

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

* When $f(x)$ is odd

$$\int_{-a}^{a} f(x)\,dx = 0$$

Examples:

1.) $\int_{-1}^{1} (x^6+1)\,dx$

$\because$ x^6+1 is an even function

$$\Rightarrow I = 2\int_{0}^{1} (x^6+1)\,dx = 2\left(\frac{x^7}{7}+x\right)_0^1$$

$$= \frac{16}{7}$$

2.) $\int_{-3}^{3} (x^3 - x)\,dx$

$\because x^3 - x$ is an odd function, i.e.,

$f(-x) = (-x)^3 + x = -x^3 + x = -(x^3 - x) = -f(x)$

$\therefore I = \int_{-3}^{3} (x^3 - x)\,dx = 0$

Integration by Substitution method:

Integration by substitution or "The Reverse chain rule" is a method to find an integral, but only when it can be set up in a special way

* The most vital step is to be able to write the given integral in the form

$$\int f(g(x))\, g'(x)\, dx = \int f(t)\, dt$$

where $t = g(x)$

Examples:

1.) $\displaystyle\int \frac{e^{\tan^{-1}x}}{1+x^2}\, dx$

let $t = \tan^{-1}x$

$\Rightarrow dt = \dfrac{1}{1+x^2}\, dx$

$\Rightarrow I = \int e^t\, dt = e^t + c$

$= e^{\tan^{-1}x} + c$

2.) $\int 2x \cos(x^2 - 5)\, dx$

let $t = x^2 - 5$

$\Rightarrow dt = 2x \cdot dx$

$\Rightarrow I = \int \cos t\, dt = \sin t + c$

$= \sin(x^2 - 5) + c$