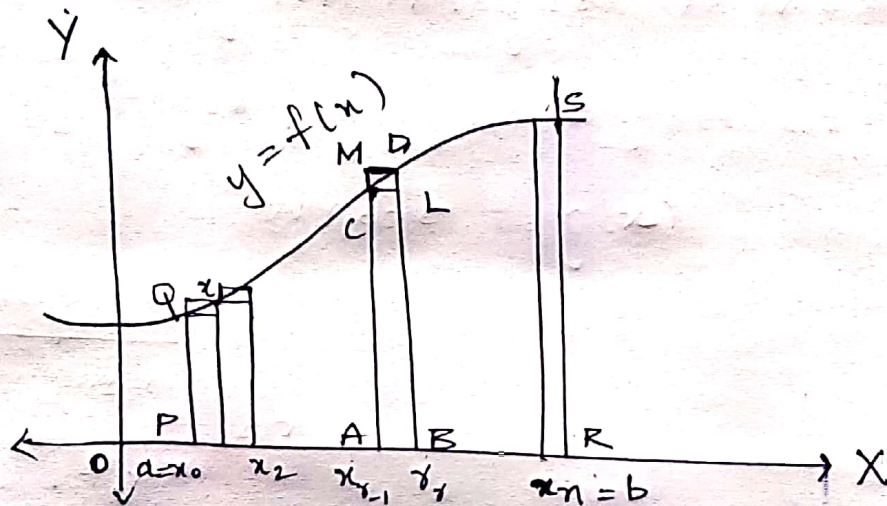


Definite Integral as the limit of sum:

Let f be a continuous function defined on the closed interval $[a, b]$. Assuming that all the values taken by the function are non-negative.

\Rightarrow the graph of the function is a curve above x -axis



The definite integral $\int_a^b f(x) dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ & the x -axis.

- To evaluate this area, consider the region PRSQP between this curve, x -axis and the ordinates $x = a$ & $x = b$

\rightarrow We now divide the interval $[a, b]$ into n equal subintervals denoted by

$$[x_0, x_1], [x_1, x_2], \dots, [x_{r-1}, x_r], \dots [x_{n-1}, x_n].$$

where $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h \dots$, $x_r = a + rh$

$$\text{So } x_n = b = a + nh \quad \text{or} \quad n = \frac{b-a}{h}$$

We note that as $n \rightarrow \infty$, $h \rightarrow 0$

The region PRSQP under consideration is the sum of n subregions, where each subregion is defined on subintervals

$$[x_{r-1}, x_r], \quad r=1, 2, 3, \dots, n$$

From the figure,

Area of the rectangle (ABLC) < Area of the region (ABDCA) < Area of the rectangle (ABDM)

①

Evidently as $x_r - x_{r-1} \rightarrow 0$

i.e., $h \rightarrow 0$ as all the three areas shown in ① become nearly equal to each other.

Now we form the following sums:

$$s_n = h[f(x_0) + \dots + f(x_{n-1})] = h \sum_{r=0}^{n-1} f(x_r)$$

s_n

②

$$S_n = h[f(x_1) + f(x_2) + \dots + f(x_n)] = h \sum_{r=1}^n f(x_r)$$

③

here,

s_n & S_n denote the sum of areas of all lower rectangles and upper rectangles raised over subintervals $[x_{r-1}, x_r]$ for $r=1, 2, 3, \dots, n$ respectively.

In view of the inequality ① for an arbitrary subinterval $[x_{r-1}, x_r]$, we have

$$s_n < \text{Area of the region} < S_n$$

PRSQP

—————→ ④

As $n \rightarrow \infty$ strips become narrower & narrower, it is assumed that the limiting values of ② & ③ are the same in both cases and the common limiting value is the required area under the curve.

Symbolically,

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} S_n = \text{area of the region PRSQP} = \int_a^b f(x) dx$$

—————→ ⑤

- It follows that this area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangles above the curve. For our convenience, we shall take rectangles with height equal to that of the curve at the left hand edge of each subinterval.

Thus, we rewrite (5) as

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h (f(a) + f(a+h) + \dots + f(a+(n-1)h))$$

(or)

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)]$$

where $h = \frac{b-a}{n} \rightarrow 0$ as $n \rightarrow \infty$

Note:

The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we choose to represent the independent variable.

$$\int_a^b f(t) dt \text{ or } \int_a^b f(u) du \text{ instead of } \int_a^b f(x) dx$$

hence, the variables t or u instead of x are called dummy variables.