Definite Integrals as limits of sums:

Question 1: (JEE Main 2019)

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+....+\frac{1}{5n}\right) \quad \text{is}$$

equal to :

(1) $\frac{\pi}{4}$ (2) $\tan^{-1}(2)$ (3) $\tan^{-1}(3)$ (4) $\frac{\pi}{2}$

Sol:

$$\lim_{x\to\infty}\sum_{r=1}^{2n}\frac{n}{n^2+r^2}$$

$$\lim_{x\to\infty}\sum_{r=1}^{2n}\frac{1}{n\left(1+\frac{r^2}{n^2}\right)}=\int_0^2\frac{dx}{1+x^2}=\tan^{-1}2$$

Question 2: (JEE Main 2019)

$$\lim_{n\to\infty}\left(\frac{(n+1)^{1/3}}{n^{4/3}}+\frac{(n+2)^{1/3}}{n^{4/3}}+......+\frac{(2n)^{1/3}}{n^{4/3}}\right) \quad \text{is}$$

equal to :

(1) $\frac{4}{3}(2)^{4/3}$

(2) $\frac{3}{4}(2)^{4/3}-\frac{4}{3}$

(3) $\frac{3}{4}(2)^{4/3}-\frac{3}{4}$

(4) $\frac{4}{3}(2)^{3/4}$

Sol:

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left(\frac{n+r}{n}\right)^{1/3}$$

$$=\int_0^1(1+x)^{1/3}dx=\frac{3}{4}\left(2^{4/3}-1\right)$$

Question 3: (JEE Advanced 2013)

For $a \in \mathbb{R}$ (the set of all real numbers), $a \neq -1$, $\lim_{n\to\infty}\frac{\left(1^a+2^a+...+n^a\right)}{(n+1)^{a-1}\left[(na+1)+(na+2)+...+(na+n)\right]}=\frac{1}{60}$.

Then a =

(A) 5

(B) 7

(C) $\frac{-15}{2}$

(D) $\frac{-17}{2}$

Sol:

$$L=\lim_{n\to\infty}\frac{1^a+2^a+\ldots\ldots+n^a}{(n+1)^{a-1}\left[\underbrace{na+na+na+\ldots+na}_{n\text{ times}}+1+2+3+\ldots+n\right]}=\lim_{n\to\infty}\frac{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^a}{n^a}\right)}{\left(\frac{n+1}{n}\right)^{a-1}\left[\frac{n^2a+\frac{n(n+1)}{2}}{n^2}\right]}$$

$$=\lim_{n\to\infty}\frac{\sum_{r=1}^{n}r^a}{(n+1)^{a-1}\left[n^2a+\frac{n(n+1)}{2}\right]}$$

$$=\lim_{n\to\infty}\frac{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^a}{n^a}\right)n^{a+1}}{(n+1)^{a-1}\left[n^2a+\frac{n(n+1)}{2}\right]}$$

$$=\frac{\int_0^1 x^a dx}{\left(a+\frac{1}{2}\right)}=\frac{1}{60}$$

$$\Rightarrow\frac{2}{(a+1)(2a+1)}=\frac{1}{60}$$

$$\Rightarrow 2a^2+3a-119=0$$

$$\Rightarrow a=7\ \&\ -\frac{17}{2}$$

$a=-\frac{17}{2}$ will be rejected as $\int_0^1 x^{-\frac{17}{2}}dx$ is not defined.