**Definite Integrals as limits of sums:**

**Question 1**

$$\lim_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)}\cdot(1^1\cdot 2^2\cdot 3^3.......n^n)^{\frac{1}{n^2}} \text{ equals}$$

(A) $\sqrt{e}$ (B) $\frac{1}{\sqrt{e}}$ (C) $\frac{1}{\sqrt[4]{e}}$ (D) $\sqrt[4]{e}$

**Solution:**

$$\text{Let } L = \lim_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)}\cdot(1^1\cdot 2^2\cdot 3^3.......n^n)^{\frac{1}{n^2}}$$

$$\ln L = \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n} k\ln k$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n}(k\ln k - k\ln n + k\ln n)$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n} k\ln\frac{k}{n} + \frac{\ln n}{n^2}\sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\ln\frac{k}{n} + \frac{\ln n}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \int_0^1 x\ln x\,dx + \frac{1}{2}\left(\frac{n+1}{n}\right)\ln n$$

$$= \int_0^1 \underbrace{x}_{II}\underbrace{\ln x}_{I}\,dx = -\frac{1}{4}; \qquad \therefore\ L = e^{-\frac{1}{4}}$$

**Question 2:**

$$\lim_{n\to\infty}\left({}^{2n}C_n\right)^{1/n}.$$

(A) 4 (B) 4/e (C) $4/e^2$ (D) $\frac{4}{e}+1$

**Solution:**

$$\text{Let } P = \lim_{n\to\infty}\left(\frac{(2n)!}{n!n!}\right)^{1/n} \quad ; \quad P = \lim_{n\to\infty}\left[\frac{n!(n+1)(n+2).....(n+n)}{n!n!}\right]^{1/n};$$

$$P = \lim_{n\to\infty}\left(\frac{n+1}{1}\cdot\frac{n+2}{2}......\frac{n+n}{n}\right)^{1/n} \qquad \ln P = \lim_{n\to\infty}\frac{1}{n}\left(\ln\frac{n+1}{1}+\ln\frac{n+2}{2}+......+\ln\frac{n+n}{n}\right);$$

$$\therefore\ T_r = \frac{1}{n}\left(\ln\frac{n+r}{r}\right)$$

$$S_n = \frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{1}{r/n}\right)$$

$$\ln P = \int_0^1 \ln\left(1+\frac{1}{x}\right)dx = \int_0^1 \left(\ln(1+x)-\ln x\right)dx$$

$$= (1+x)\ln(1+x) - (1+x) - [x\ln x - x]$$

$$= [(1+x)\ln(1+x) - 1 - x\ln x]_0^1 = (2\ln 2 - 1 - 0) - (0 - 1)$$

$$\ln P = \ln 4 \Rightarrow \boxed{P = 4}$$

**Question 3:**

Evaluate

$$\lim_{n\to\infty}\frac{[(n+1)(n+2).....(n+n)]^{1/n}}{n}$$

**Solution:**

Let $L$ represent the given limit. We have,

$$\ln L = \frac{1}{n}\ln\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)......\left(1+\frac{n}{n}\right)\right\}$$
$$= \frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{r}{n}\right)$$

**Since,** $$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{r=1}^{n} hf(a+rh)$$ **_____ (1)**

$$where \quad h = \frac{b-a}{n}$$

$$\begin{aligned}\ln L &= \int_0^1 \ln(1+x)dx \\ &= x\ln(1+x)|_0^1 - \int_0^1 \frac{x}{1+x}dx \\ &= \ln 2 - \int_0^1 \left(\frac{x+1-1}{x+1}\right)dx \\ &= \ln 2 - 1 + \ln 2 \\ &= 2\ln 2 - 1 \\ &= \ln(4/e)\end{aligned}$$

$$\Rightarrow \quad \mathbf{L} = \frac{4}{e}$$

**Question 4:**

Find the sum of the series

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{3n} \text{as } n \to \infty$$

**Solution:** The given series can be written concisely as

$$\begin{aligned}S &= \sum_{r=0}^{2n} \frac{1}{n+r} \\ &= \frac{1}{n}\sum_{r=0}^{2n} \frac{1}{1+(r/n)}\end{aligned}$$

Comparing this with the right hand side of (1), we see that $S$ can be expressed as the integral of a function $f(x) = \frac{1}{1+x}$ from 0 to 2, because since $r$ varies till $2n$, $r/n$ varies till 2. Thus,

$$\begin{aligned}S &= \int_0^2 \frac{1}{1+x}dx \\ &= \ln 3\end{aligned}$$

**Question 5:**

Find the sum of the series

$$\frac{n}{n^2+1^2}+\frac{1}{n^2+2^2}+\ldots\ldots+\frac{1}{n^2+n^2} \text{ as } n\to\infty$$

**Solution:**

Concisely put,

$$\begin{aligned} S &= \sum_{l=1}^{n}\frac{n}{n^2+r^2} \\ &= \frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+(r/n)^2} \end{aligned}$$

As discussed earlier, $S$ can be written in integral form as

$$\begin{aligned} S &= \int_0^1 \frac{1}{1+x^2}dx \\ &= \tan^{-1}x\Big|_0^1 \\ &= \frac{\pi}{4} \end{aligned}$$