Integration by substitution method:

Question 1: (JEE Main 2015)

The integral $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ equals :

(1) $(x^4+1)^{1/4}+c$

(2) $-(x^4+1)^{1/4}+c$

(3) $-\left(\frac{x^4+1}{x^4}\right)^{1/4}+c$

(4) $\left(\frac{x^4+1}{x^4}\right)^{1/4}+c$

Sol: $\int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$ $\Rightarrow 1+\frac{1}{x^4}=t$

$$\Rightarrow \frac{-4}{x^5}dx = dt$$

$$\Rightarrow -\frac{1}{4}\int \frac{1}{t^{3/4}}dt$$

$$= -\frac{1}{4}\times 4t^{1/4}+c = -\left(1+\frac{1}{x^4}\right)^{1/4}+c$$

Question 2: (JEE Main 2016)

The integral $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$ is equal to :-

(1) $\frac{-x^{10}}{2(x^5+x^3+1)^2}+C$

(2) $\frac{-x^5}{(x^5+x^3+1)^2}+C$

(3) $\frac{x^{10}}{2(x^5+x^3+1)^2}+C$

(4) $\frac{x^5}{2(x^5+x^3+1)^2}+C$

Sol:

(3)

÷ by x^{15} in N^r & D^r

$$\int \frac{\left(\frac{2}{x^3}+\frac{5}{x^6}\right)dx}{\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^3}$$

Let $1 + \frac{1}{x^2} + \frac{1}{x^5} = t \Rightarrow dt = -\left(\frac{2}{x^3}+\frac{5}{x^6}\right)dx$

$$\int \frac{-dt}{t^3} = \frac{1}{2t^2}+c$$

Question 3: (JEE Main 2018)

The integral $\int \frac{\sin^2 x\cos^2 x}{\left(\sin^5 x+\cos^3 x\sin^2 x+\sin^3 x\cos^2 x+\cos^5 x\right)^2}dx$ is equal to :

(1) $\frac{-1}{1+\cot^3 x}+C$

(2) $\frac{1}{3\left(1+\tan^3 x\right)}+C$

(3) $\frac{-1}{3\left(1+\tan^3 x\right)}+C$

(4) $\frac{1}{1+\cot^3 x}+C$

Sol:

$$\int \frac{\sin^2 x\cos^2 x}{\left(\sin^5 x+\cos^3 x\sin^2 x+\sin^3 x\cos^2 x+\cos^5 x\right)^2}dx$$

$$= \int \frac{\sin^2 x\cos^2 x}{\left(\sin^3 x+\cos^3 x\right)^2}dx$$

$$= \int \frac{\tan^2 x\cdot\sec^2 x}{\left(1+\tan^3 x\right)^2}dx$$

put $1 + \tan^3 x = t$

$$= \frac{1}{3}\int \frac{dt}{t^2} \Rightarrow -\frac{1}{3t}+c$$

$$= -\frac{1}{3\left(1+\tan^3 x\right)}+c$$

Question 4: (JEE Main 2019)

If $f(x)=\int \frac{5x^8+7x^6}{\left(x^2+1+2x^7\right)^2}dx,(x\geq 0)$ and $f(0)=0$, then the value of $f(1)$ is :

(1) $-\frac{1}{2}$ (2) $\frac{1}{2}$ (3) $-\frac{1}{4}$ (4) $\frac{1}{4}$

Sol :

$$\int \frac{5x^8+7x^6}{\left(x^2+1+2x^7\right)^2}dx$$

$$=\int \frac{5x^{-6}+7x^{-8}}{\left(\frac{1}{x^7}+\frac{1}{x^5}+2\right)^2}dx=\frac{1}{2+\frac{1}{x^5}+\frac{1}{x^7}}+C$$

As $f(0)=0,\ f(x)=\frac{x^7}{2x^7+x^2+1}$

$f(1)=\frac{1}{4}$

Question 5: (JEE Main 2019)

Let $n\geq 2$ be a natural number and $0<\theta<\pi/2$.

Then $\int \frac{(\sin^n\theta-\sin\theta)^{\frac{1}{n}}\cos\theta}{\sin^{n+1}\theta}d\theta$ is equal to :

(Where C is a constant of integration)

(1) $\frac{n}{n^2-1}\left(1-\frac{1}{\sin^{n+1}\theta}\right)^{\frac{n+1}{n}}+C$

(2) $\frac{n}{n^2+1}\left(1-\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+C$

(3) $\frac{n}{n^2-1}\left(1-\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+C$

(4) $\frac{n}{n^2-1}\left(1+\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+C$

Sol:

$$\int \frac{(\sin^n\theta-\sin\theta)^{1/n}\cos\theta}{\sin^{n+1}\theta}d\theta$$

$$=\int \frac{\sin\theta\left(1-\frac{1}{\sin^{n-1}\theta}\right)^{1/n}}{\sin^{n+1}\theta}d\theta$$

Put $1-\frac{1}{\sin^{n-1}\theta}=t$

So $\frac{(n-1)}{\sin^n\theta}\cos\theta d\theta=dt$

Now $\frac{1}{n-1}\int (t)^{1/n}dt$

$$=\frac{1}{(n-1)}\frac{(t)^{\frac{1}{n}+1}}{\frac{1}{n}+1}+C$$

$$=\frac{1}{(n-1)}\left(1-\frac{1}{\sin^{n-1}\theta}\right)^{\frac{1}{n}+1}+C$$

Question 6: (JEE Main 2019)

If $\int \frac{x+1}{\sqrt{2x-1}}dx = f(x)\sqrt{2x-1}+C$, where C is a constant of integration, then f(x) is equal to :-

(1) $\frac{1}{3}(x+4)$ (2) $\frac{1}{3}(x+1)$

(3) $\frac{2}{3}(x+2)$ (4) $\frac{2}{3}(x-4)$

Sol:

$$\sqrt{2x-1}=t \Rightarrow 2x-1=t^2 \Rightarrow 2dx = 2t.dt$$

$$\int \frac{x+1}{\sqrt{2x-1}}dx = \int \frac{\frac{t^2+1}{2}+1}{t}t\,dt = \int \frac{t^2+3}{2}dt$$

$$=\frac{1}{2}\left(\frac{t^3}{3}+3t\right)=\frac{t}{6}\left(t^2+9\right)+c$$

$$=\sqrt{2x-1}\left(\frac{2x-1+9}{6}\right)+c=\sqrt{2x-1}\left(\frac{x+4}{3}\right)+c$$

$$\Rightarrow f(x)=\frac{x+4}{3}$$

Question 7: (JEE Main 2019)

The integral $\int_1^e \left\{\left(\frac{x}{e}\right)^{2x}-\left(\frac{e}{x}\right)^x\right\}\log_e x\,dx$ is equal to :

(1) $\frac{1}{2}-e-\frac{1}{e^2}$ (2) $\frac{3}{2}-\frac{1}{e}-\frac{1}{2e^2}$

(3) $-\frac{1}{2}+\frac{1}{e}-\frac{1}{2e^2}$ (4) $\frac{3}{2}-e-\frac{1}{2e^2}$

Sol:

$$\int_1^e \left(\frac{x}{e}\right)^{2x} \log_e x.dx - \int_1^e \left(\frac{e}{x}\right) \log_e x.dx$$

$$\text{Let } \left(\frac{x}{e}\right)^{2x} = t,\ \left(\frac{e}{x}\right)^{x} = v$$

$$= \frac{1}{2}\int_{\left(\frac{1}{e}\right)^2}^{1} dt \ + \int_e^1 dv$$

$$= \frac{1}{2}\left(1-\frac{1}{e^2}\right) + (1-e) = \frac{3}{2} - \frac{1}{2e^2} - e$$

Question 8: (JEE Main 2019)

The integral $\int_{\pi/6}^{\pi/3} \sec^{2/3} x \operatorname{cosec}^{4/3} x \, dx$ equal to:

(1) $3^{7/6} - 3^{5/6}$ (2) $3^{5/3} - 3^{1/3}$

(3) $3^{4/3} - 3^{1/3}$ (4) $3^{5/6} - 3^{2/3}$

Sol:

$$I = \int \frac{1}{\cos^{2/3} x \sin^{1/3} x . \sin x} dx$$

$$= \int \frac{\tan^{2/3} x}{\tan^2 x} . \sec^2 x . dx$$

$$= \int \frac{\sec^2 x}{\tan^{4/3} x} . dx \qquad \{\tan x = t,\ \sec^2 x dx = dt\}$$

$$= \int \frac{dt}{t^{4/3}} = \frac{t^{-1/3}}{-1/3} = -3\left(t^{-1/3}\right)$$

$$\Rightarrow I = -3\tan(x)^{-1/3}$$

$$\Rightarrow I = \frac{3}{(\tan x)^{1/3}}\Bigg|_{\pi/6}^{\pi/3} = -3\left(\frac{1}{\left(\sqrt{3}\right)^{1/3}} - \left(\sqrt{3}\right)^{1/3}\right)$$

$$= 3\left(3^{1/3} - \frac{1}{3^{1/6}}\right) = 3^{7/6} - 3^{5/6}$$

Question 9: (JEE Main 2020)

The integral $\int \frac{dx}{(x+4)^{\frac{8}{7}}(x-3)^{\frac{6}{7}}}$ is equal to :

(where C is a constant of integration)

(1) $\left(\frac{x-3}{x+4}\right)^{\frac{1}{7}}+C$ (2) $-\left(\frac{x-3}{x+4}\right)^{-\frac{1}{7}}+C$

(3) $\frac{1}{2}\left(\frac{x-3}{x+4}\right)^{\frac{3}{7}}+C$ (4) $-\frac{1}{13}\left(\frac{x-3}{x+4}\right)^{-\frac{13}{7}}+C$

Sol:

$$I=\int \frac{dx}{(x+4)^{\frac{8}{7}}(x-3)^{\frac{6}{7}}}=\int \frac{dx}{\left(\frac{x+4}{x-3}\right)^{\frac{8}{7}}(x-3)^{2}}$$

Let $\frac{x+4}{x-3}=t \Rightarrow \frac{dx}{(x-3)^2}=-\frac{1}{7}dt$

$$\Rightarrow \quad I=-\frac{1}{7}\int \frac{dt}{t^{8/7}}=-\frac{1}{7}\int t^{-8/7}dt$$

$$=t^{-1/7}+C=+\left(\frac{x+4}{x-3}\right)^{-1/7}+C=\left(\frac{x-3}{x+4}\right)^{1/7}+C$$

Question 10: (JEE Main 2020)

Let $f(x)=\int \frac{\sqrt{x}}{(1+x)^2}dx\ (x \geq 0)$. Then $f(3)-f(1)$

is equal to :

(1) $-\frac{\pi}{6}+\frac{1}{2}+\frac{\sqrt{3}}{4}$ (2) $\frac{\pi}{6}+\frac{1}{2}-\frac{\sqrt{3}}{4}$

(3) $-\frac{\pi}{12}+\frac{1}{2}+\frac{\sqrt{3}}{4}$ (4) $\frac{\pi}{12}+\frac{1}{2}-\frac{\sqrt{3}}{4}$

Sol:

$$f(x) = \int_1^3 \frac{\sqrt{x}\,dx}{(1+x)^2} = \int_1^{\sqrt{3}} \frac{t.2t\,dt}{(1+t^2)^2} \quad \left(\text{put } \sqrt{x}=t\right)$$

$$= \left(-\frac{t}{1+t^2}\right)_1^{\sqrt{3}} + (\tan^{-1} t)_1^{\sqrt{3}} \quad \text{[Appling by parts]}$$

$$= -\left(\frac{\sqrt{3}}{4}-\frac{1}{2}\right)+\frac{\pi}{3}-\frac{\pi}{4}$$

$$= \frac{1}{2}-\frac{\sqrt{3}}{4}+\frac{\pi}{12}$$

Question 11: (JEE Main 2020)

If $\int \frac{\cos\theta}{5+7\sin\theta-2\cos^2\theta}\,d\theta = A \log_e |B(\theta)| + C$,

where C is a constant of integration, then $\frac{B(\theta)}{A}$

can be :

(1) $\frac{2\sin\theta+1}{5(\sin\theta+3)}$ (2) $\frac{2\sin\theta+1}{\sin\theta+3}$

(3) $\frac{5(\sin\theta+3)}{2\sin\theta+1}$ (4) $\frac{5(2\sin\theta+1)}{\sin\theta+3}$

Sol:

$$\int \frac{\cos\theta\ d\theta}{5+7\sin\theta-2\cos^2\theta}$$

$$\int \frac{\cos\theta\,d\theta}{3+7\sin\theta+2\sin^2\theta} \quad \boxed{\begin{array}{l}\sin\theta=t\\ \cos\theta d\theta=dt\end{array}}$$

$$\int \frac{dt}{2t^2+7t+3} = \int \frac{dt}{(2t+1)(t+3)}$$

$$= \frac{1}{5}\int\left(\frac{2}{2t+1}-\frac{1}{t+3}\right)dt$$

$$= \frac{1}{5} ln\left|\frac{2t+1}{t+3}\right|+C$$

$$= \frac{1}{5} ln\left|\frac{2\sin\theta+1}{\sin\theta+3}\right|+C$$

$$A = \frac{1}{5} \text{ and } B(\theta) = \frac{2\sin\theta+1}{\sin\theta+3}$$