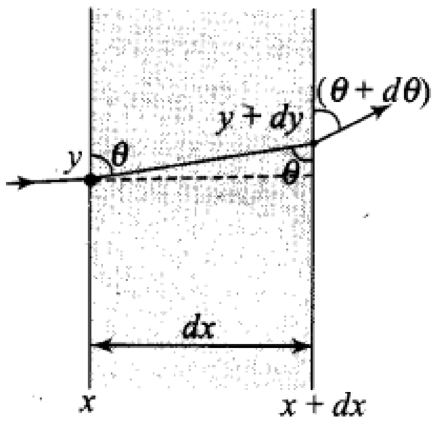
Question 6. The mixture of a pure liquid and a solution in a long vertical column (i.e., horizontal dimensions << vertical dimensions) produces diffusion of solute particles and hence a refractive index gradient along the vertical dimension. A ray of light entering the column at right angles to the vertical is deviated from its original path. Find the deviation in travelling a horizontal distance d << h, the height of the column.

Solution:

Let us consider a portion of a ray between x and $x + dx$ inside the liquid solution. Let the angle of incidence of ray at x be θ and let the ray enters the thin column at height y. Because of the refraction it deviates from the original path and emerges at $x + dx$ with an angle $\theta + d\theta$ and at a height $y + dy$.



From Snell's law,

$$\mu(y) \sin \theta = \mu(y + dy) \sin (\theta + d\theta)) \quad \ldots\text{(i)}$$

Let refractive index of the liquid at position y be $\mu(y) = \mu$, then

$$\mu(y + dy) = \mu + \left(\frac{d\mu}{dy}\right) dy = \mu + kdy$$

where $k = \left(\frac{d\mu}{dy}\right)$ = refractive index gradient along the vertical dimension.

Hence from (i), $\mu \sin \theta = (\mu + kdy) \cdot \sin (\theta + d\theta)$

$$\mu \sin \theta = (\mu + kdy) \cdot (\sin \theta \cdot \cos d\theta + \cos \theta \cdot \sin d\theta)$$

$$\mu \sin \theta = (\mu + kdy) \cdot (\sin \theta \cdot 1 + \cos \theta \cdot d\theta) \quad \ldots\text{(ii)}$$

For small angle $\sin d\theta \approx d\theta$ and $\cos d\theta \approx 1$

$$\mu \sin \theta = \mu \sin \theta + kdy \sin \theta + \mu \cos \theta \cdot d\theta + k \cos \theta dy \cdot d\theta$$

$$kdy \sin \theta + \mu \cos \theta \cdot d\theta = 0 \Rightarrow d\theta = -\frac{k}{\mu} \tan \theta \, dy$$

But $\tan\theta = \frac{dx}{dy}$ and $k = \left(\frac{d\mu}{dy}\right)$

$$d\theta = -\frac{k}{\mu}\left(\frac{dx}{dy}\right)dy \Rightarrow d\theta = -\frac{k}{\mu}dx$$

Integrating both sides, $\int_0^{\delta} d\theta = -\frac{k}{\mu}\int_0^{d} dx$

$$\Rightarrow \quad \delta = -\frac{kd}{\mu} = -\frac{d}{\mu}\left(\frac{d\mu}{dy}\right)$$