Pythagoras theorem, BC = $2\sqrt{2}$ m. The forces on the ladder are its weight W acting at its centre of gravity D, reaction forces F_1 and F_2 of the wall and the floor respectively. Force F_1 is perpendicular to the wall, since the wall is frictionless. Force F_2 is resolved into two components, the normal reaction N and the force of friction F. Note that F prevents the ladder from sliding away from the wall and is therefore directed toward the wall.

For translational equilibrium, taking the forces in the vertical direction,

$$N - W = 0 \qquad \text{(i)}$$

Taking the forces in the horizontal direction,

$$F - F_1 = 0 \qquad \text{(ii)}$$

For rotational equilibrium, taking the moments of the forces about A,

$$2\sqrt{2}\,F_1 - (1/2)\,W = 0 \qquad \text{(iii)}$$

Now $W = 20\text{ g} = 20 \times 9.8\text{ N} = 196.0\text{ N}$

From (i) $N = 196.0\text{ N}$

From (iii) $F_1 = W/4\sqrt{2} = 196.0/4\sqrt{2} = 34.6\text{ N}$

From (ii) $F = F_1 = 34.6\text{ N}$

$$F_2 = \sqrt{F^2 + N^2} = 199.0\text{ N}$$

The force F_2 makes an angle α with the horizontal,

$$\tan\alpha = N/F = 4\sqrt{2}\,, \quad \alpha = \tan^{-1}(4\sqrt{2}) \approx 80^\circ$$ ◀

7.9 MOMENT OF INERTIA

We have already mentioned that we are developing the study of rotational motion parallel to the study of translational motion with which we are familiar. We have yet to answer one major question in this connection. **What is the analogue of mass in rotational motion?** We shall attempt to answer this question in the present section. To keep the discussion simple, we shall consider rotation about a fixed axis only. Let us try to get an expression for *the* **kinetic energy of a rotating body**. We know that for a body rotating about a fixed axis, each particle of the body moves in a circle with linear velocity given by Eq. (7.19). (Refer to Fig. 7.16). For a particle at a distance from the axis, the linear velocity is $\upsilon_i = r_i\omega$. The kinetic energy of motion of this particle is

$$k_i = \frac{1}{2}m_i\upsilon_i^2 = \frac{1}{2}m_i r_i^2\omega^2$$

where m_i is the mass of the particle. The total kinetic energy K of the body is then given by the sum of the kinetic energies of individual particles,

$$K = \sum_{i=1}^{n} k_i = \frac{1}{2}\sum_{i=1}^{n}(m_i r_i^2\omega^2)$$

Here n is the number of particles in the body. Note ω is the same for all particles. Hence, taking ω out of the sum,

$$K = \frac{1}{2}\omega^2\Big(\sum_{i=1}^{n} m_i r_i^2\Big)$$

We define a new parameter characterising the rigid body, called the moment of inertia I, given by

$$I = \sum_{i=1}^{n} m_i r_i^2 \qquad (7.34)$$

With this definition,

$$K = \frac{1}{2}I\omega^2 \qquad (7.35)$$

Note that the parameter I is independent of the magnitude of the angular velocity. It is a characteristic of the rigid body and the axis about which it rotates.

Compare Eq. (7.35) for the kinetic energy of a rotating body with the expression for the kinetic energy of a body in linear (translational) motion,

$$K = \frac{1}{2}m\,\upsilon^2$$

Here, m is the mass of the body and υ is its velocity. We have already noted the analogy between angular velocity ω (in respect of rotational motion about a fixed axis) and linear velocity υ (in respect of linear motion). It is then evident that the parameter, moment of inertia I, is the desired rotational analogue of mass in linear motion. In rotation (about a fixed axis), the moment of inertia plays a similar role as mass does in linear motion.

We now apply the definition Eq. (7.34), to calculate the moment of inertia in two simple cases.

(a) Consider a thin ring of radius R and mass M, rotating in its own plane around its centre with angular velocity ω. Each mass element of the ring is at a distance R from the axis, and moves with a speed $R\omega$. The kinetic energy is therefore,

$$K = \frac{1}{2}Mv^2 = \frac{1}{2}MR^2\omega^2$$

Comparing with Eq. (7.35) we get $I = MR^2$ for the ring.

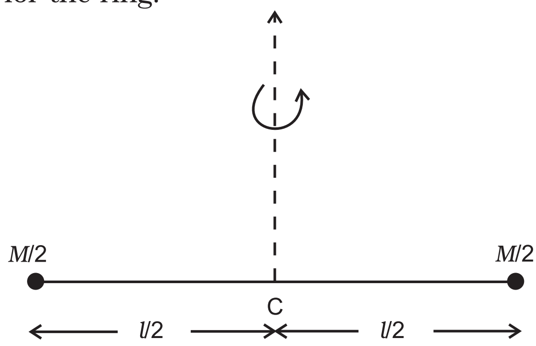


Fig. 7.28 *A light rod of length l with a pair of masses rotating about an axis through the centre of mass of the system and perpendicular to the rod. The total mass of the system is M.*

(b) Next, take a rigid rod of negligible mass of length of length l with a pair of small masses, rotating about an axis through the centre of mass perpendicular to the rod (Fig. 7.28). Each mass $M/2$ is at a distance $l/2$ from the axis. The moment of inertia of the masses is therefore given by

$$(M/2)(l/2)^2 + (M/2)(l/2)^2$$

Thus, for the pair of masses, rotating about the axis through the centre of mass perpendicular to the rod

$$I = Ml^2 / 4$$

Table 7.1 simply gives the moment of inertia of various familiar regular shaped bodies about specific axes. (The derivations of these expressions are beyond the scope of this textbook and you will study them in higher classes.)

As the mass of a body resists a change in its state of linear motion, it is a measure of its inertia in linear motion. Similarly, as the moment of inertia about a given axis of rotation resists a change in its rotational motion, it can be regarded as a measure of rotational inertia of the body; it is a measure of the way in which different parts of the body are distributed at different distances from the axis. Unlike the mass of a body, the moment of inertia is not a fixed quantity but depends on distribution of mass about the axis of rotation, and the orientation and position of the axis of rotation with respect to the body as a whole. As a measure of the way in which the mass of a rotating rigid body is distributed with respect to the axis of rotation, we can define a new parameter, the **radius of gyration**. It is related to the moment of inertia and the total mass of the body.

Notice from the Table 7.1 that in all cases, we can write $I = Mk^2$, where k has the dimension of length. For a rod, about the perpendicular axis at its midpoint, $k^2 = L^2/12$, i.e. $k = L/\sqrt{12}$. Similarly, $k = R/2$ for the circular disc about its diameter. The length k is a geometric property of the body and axis of rotation. It is called the **radius of gyration**. **The radius of gyration of a body about an axis** may be defined as the distance from the axis of a mass point whose mass is equal to the mass of the whole body and whose moment of inertia is equal to the moment of inertia of the body about the axis.

Thus, the moment of inertia of a rigid body depends on the mass of the body, its shape and size; distribution of mass about the axis of rotation, and the position and orientation of the axis of rotation.

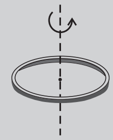
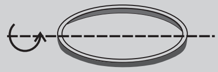
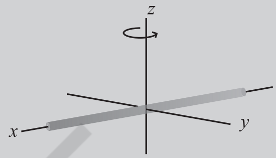
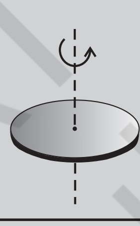
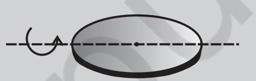
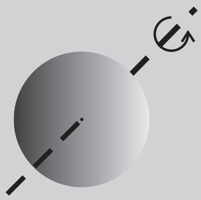
From the definition, Eq. (7.34), we can infer that the dimensions of moments of inertia are ML^2 and its SI units are kg m^2.

The property of this extremely important quantity I, as a measure of rotational inertia of the body, has been put to a great practical use. The machines, such as steam engine and the automobile engine, etc., that produce rotational motion have a disc with a large moment of inertia, called a **flywheel**. Because of its large moment of inertia, the flywheel resists the sudden increase or decrease of the speed of the vehicle. It allows a gradual change in the speed and prevents jerky motions, thereby ensuring a smooth ride for the passengers on the vehicle.

7.10 THEOREMS OF PERPENDICULAR AND PARALLEL AXES

These are two useful theorems relating to moment of inertia. We shall first discuss the theorem of perpendicular axes and its simple yet instructive application in working out the moments of inertia of some regular-shaped bodies.

Table 7.1 Moments of inertia of some regular shaped bodies about specific axes

Z	Body	Axis	Figure	I
(1)	Thin circular ring, radius R	Perpendicular to plane, at centre		$M R^2$
(2)	Thin circular ring, radius R	Diameter		$M R^2/2$
(3)	Thin rod, length L	Perpendicular to rod, at mid point		$M L^2/12$
(4)	Circular disc, radius R	Perpendicular to disc at centre		$M R^2/2$
(5)	Circular disc, radius R	Diameter		$M R^2/4$
(6)	Hollow cylinder, radius R	Axis of cylinder		$M R^2$
(7)	Solid cylinder, radius R	Axis of cylinder		$M R^2/2$
(8)	Solid sphere, radius R	Diameter		$2 M R^2/5$

Theorem of perpendicular axes

This theorem is applicable to bodies which are planar. In practice this means the theorem applies to flat bodies whose thickness is very small compared to their other dimensions (e.g. length, breadth or radius). Fig. 7.29 illustrates the theorem. It states that **the moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.**

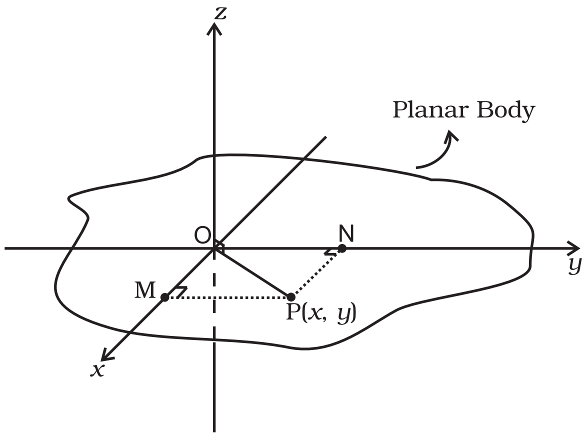


Fig. 7.29 *Theorem of perpendicular axes applicable to a planar body; x and y axes are two perpendicular axes in the plane and the z-axis is perpendicular to the plane.*

The figure shows a planar body. An axis perpendicular to the body through a point O is taken as the *z*-axis. Two mutually perpendicular axes lying in the plane of the body and concurrent with *z*-axis, i.e., passing through O, are taken as the *x* and *y*-axes. The theorem states that

$$I_z = I_x + I_y \qquad (7.36)$$

Let us look at the usefulness of the theorem through an example.

Example 7.10 What is the moment of inertia of a disc about one of its diameters?

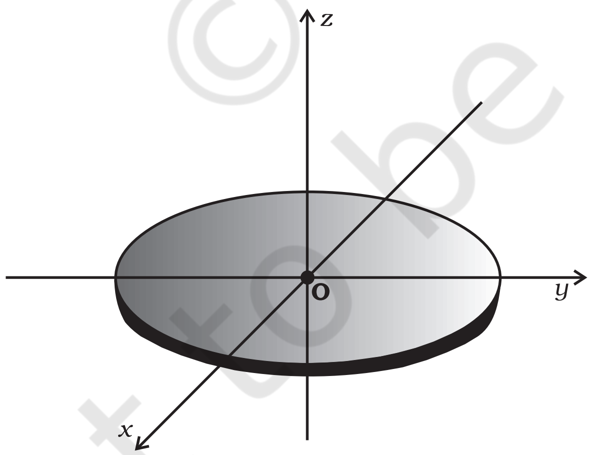


Fig. 7.30 *Moment of inertia of a disc about a diameter, given its moment of inertia about the perpendicular axis through its centre.*

Answer We assume the moment of inertia of the disc about an axis perpendicular to it and through its centre to be known; it is $MR^2/2$, where M is the mass of the disc and *R* is its radius (Table 7.1)

The disc can be considered to be a planar body. Hence the theorem of perpendicular axes is applicable to it. As shown in Fig. 7.30, we take three concurrent axes through the centre of the disc, O, as the *x*–, *y*– and *z*–axes; *x*– and *y*–axes lie in the plane of the disc and *z*–axis is perpendicular to it. By the theorem of perpendicular axes,

$$I_z = I_x + I_y$$

Now, *x* and *y* axes are along two diameters of the disc, and by symmetry the moment of inertia of the disc is the same about any diameter. Hence

$$I_x = I_y$$

and $\quad I_z = 2I_x$

But $\quad I_z = MR^2/2$

So finally, $\quad I_x = I_z/2 = MR^2/4$

Thus the moment of inertia of a disc about any of its diameter is $MR^2/4$. ◀

Find similarly the moment of inertia of a ring about any of its diameters. Will the theorem be applicable to a solid cylinder?

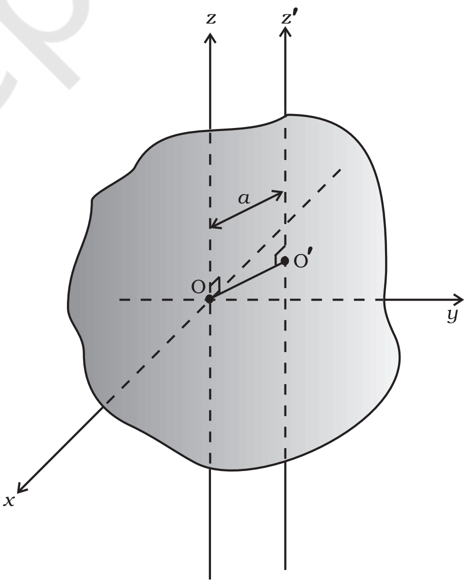


Fig.7.31 *The theorem of parallel axes The z and z′ axes are two parallel axes separated by a distance a; O is the centre of mass of the body, OO′ = a.*

7.10.1 Theorem of parallel axes

This theorem is applicable to a body of any shape. It allows to find the moment of inertia of a body about any axis, given the moment of inertia of the body about a parallel axis through the centre of mass of the body. We shall only state this theorem and not give its proof. We shall, however, apply it to a few simple situations which will be enough to convince us about the usefulness of the theorem. The theorem may be stated as follows:

The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes. As shown in the Fig. 7.31, z and z' are two parallel axes, separated by a distance a. The z-axis passes through the centre of mass O of the rigid body. Then according to the theorem of parallel axes

$$I_{z'} = I_z + Ma^2 \qquad (7.37)$$

where I_z and $I_{z'}$ are the moments of inertia of the body about the z and z' axes respectively, M is the total mass of the body and a is the perpendicular distance between the two parallel axes.

Example 7.11 What is the moment of inertia of a rod of mass M, length l about an axis perpendicular to it through one end?

Answer For the rod of mass M and length l, $I = Ml^2/12$. Using the parallel axes theorem, $I' = I + Ma^2$ with $a = l/2$ we get,

$$I' = M\frac{l^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$$

We can check this independently since I is half the moment of inertia of a rod of mass $2M$ and length $2l$ about its midpoint,

$$I' = 2M.\frac{4l^2}{12} \times \frac{1}{2} = \frac{Ml^2}{3}$$

Example 7.12 What is the moment of inertia of a ring about a tangent to the circle of the ring?

Answer

The tangent to the ring in the plane of the ring is parallel to one of the diameters of the ring. The distance between these two parallel axes is R, the radius of the ring. Using the parallel axes theorem,

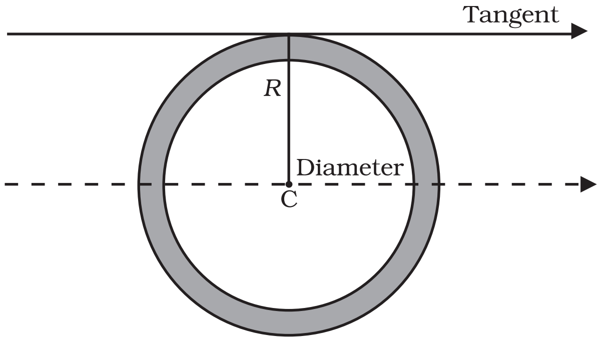


Fig. 7.32

$$I_{tangent} = I_{dia} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2.$$

7.11 KINEMATICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

We have already indicated the analogy between rotational motion and translational motion. For example, the angular velocity $\boldsymbol{\omega}$ plays the same role in rotation as the linear velocity $\mathbf{v}$ in translation. We wish to take this analogy further. In doing so we shall restrict the discussion only to rotation about fixed axis. This case of motion involves only one degree of freedom, i.e., needs only one independent variable to describe the motion. This in translation corresponds to linear motion. This section is limited only to kinematics. We shall turn to dynamics in later sections.

We recall that for specifying the angular displacement of the rotating body we take any particle like P (Fig.7.33) of the body. Its angular displacement θ in the plane it moves is the angular displacement of the whole body; θ is measured from a fixed direction in the plane of motion of P, which we take to be the x'-axis, chosen parallel to the x-axis. Note, as shown, the axis of rotation is the z – axis and the plane of the motion of the particle is the x - y plane. Fig. 7.33 also shows θ_0, the angular displacement at $t = 0$.

We also recall that the angular velocity is the time rate of change of angular displacement, $\omega = d\theta/dt$. Note since the axis of rotation is fixed,