Chapter 8

BINOMIAL THEOREM

8.1 Overview:

8.1.1 An expression consisting of two terms, connected by + or – sign is called a binomial expression. For example, $x+a$, $2x-3y$, $\frac{1}{x}-\frac{1}{x^3}$, $7x-\frac{4}{5y}$, etc., are all binomial expressions.

8.1.2 ***Binomial theorem***

If a and b are real numbers and n is a positive integer, then

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots$$

$$\ldots + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_n\, b^n, \text{ where } {}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r} \text{ for } 0 \le r \le n$$

The general term or $(r+1)^{\text{th}}$ term in the expansion is given by

$$T_{r+1} = {}^nC_r\, a^{n-r} b^r$$

8.1.3 ***Some important observations***

1. The total number of terms in the binomial expansion of $(a+b)^n$ is $n+1$, i.e. one more than the exponent n.
2. In the expansion, the first term is raised to the power of the binomial and in each subsequent terms the power of a reduces by one with simultaneous increase in the power of b by one, till power of b becomes equal to the power of binomial, i.e., the power of a is n in the first term, $(n-1)$ in the second term and so on ending with zero in the last term. At the same time power of b is 0 in the first term, 1 in the second term and 2 in the third term and so on, ending with n in the last term.
3. In any term the sum of the indices (exponents) of 'a' and 'b' is equal to n (i.e., the power of the binomial).
4. The coefficients in the expansion follow a certain pattern known as pascal's triangle.

Index of Binomial	Coefficient of various terms										
0						1					
1					1		1				
2				1		2		1			
3			1		3		3		1		
4		1		4		6		4		1	
5	1		5		10		10		5		1

Each coefficient of any row is obtained by adding two coefficients in the preceding row, one on the immediate left and the other on the immediate right and each row is bounded by 1 on both sides.

The $(r+1)^{th}$ term or general term is given by

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

8.1.4 *Some particular cases*

If n is a positive integer, then

$$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^n b^1 + {}^nC_2\, a^{n-2}\, b^2 + \ldots + {}^nC_r\, a^{n-r}\, b^r + \ldots + {}^nC_n\, a^0 b^n \qquad \ldots (1)$$

In particular

1. Replacing b by $-b$ in (i), we get

$$(a-b)^n = {}^nC_0\, a^n b^0 - {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots + (-1)^r\, {}^nC_r\, a^{n-r} b^r + \ldots + (-1)^n\, {}^nC_n\, a^0 b^n \qquad \ldots (2)$$

2. Adding (1) and (2), we get

$$(a+b)^n + (a-b)^n = 2\,[{}^nC_0\, a^n b^0 + {}^nC_2\, a^{n-2} b^2 + {}^nC_4\, a^{n-4} b^4 + \ldots]$$
$$= 2\,[\text{terms at odd places}]$$

3. Subtracting (2) from (1), we get

$$(a+b)^n - (a-b)^n = 2\,[{}^nC_1\, a^{n-1} b^1 + {}^nC_3\, a^{n-3} b^3 + \ldots]$$
$$= 2\,[\text{sum of terms at even places}]$$

4. Replacing a by 1 and b by x in (1), we get

$$(1+x)^n = {}^nC_0\, x^0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_{n-1}\, x^{n-1} + {}^nC_n\, x^n$$

i.e. $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

5. Replacing a by 1 and b by $-x$ in ... (1), we get

$$(1-x)^n = {}^nC_0\, x^0 - {}^nC_1\, x + {}^nC_2\, x^2 \; ... + {}^nC_{n-1}\, (-1)^{n-1}\, x^{n-1} + {}^nC_n\, (-1)^n\, x^n$$

i.e., $$(1-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^r$$

8.1.5 *The p^{th} term from the end*

The p^{th} term from the end in the expansion of $(a+b)^n$ is $(n-p+2)^{th}$ term from the beginning.

8.1.6 *Middle terms*

The middle term depends upon the value of n.

(a) If n is even: then the total number of terms in the expansion of $(a+b)^n$ is $n+1$ (odd). Hence, there is only one middle term, i.e., $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term.

(b) If n is odd: then the total number of terms in the expansion of $(a+b)^n$ is $n+1$ (even). So there are two middle terms i.e., $\frac{n+1}{2}^{th}$ and $\frac{n+3}{2}^{th}$ are two middle terms.

8.1.7 *Binomial coefficient*

In the Binomial expression, we have

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}\, b + {}^nC_2\, a^{n-2}\, b^2 + ... + {}^nC_n\, b^n \qquad ...(1)$$

The coefficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ... , ${}^nC_n$ are known as binomial or combinatorial coefficients.

Putting $a = b = 1$ in (1), we get

$$ {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$$

Thus the sum of all the binomial coefficients is equal to 2^n.

Again, putting $a = 1$ and $b = -1$ in (i), we get

$$ {}^nC_0 + {}^nC_2 + {}^nC_4 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ...$$

Thus, the sum of all the odd binomial coefficients is equal to the sum of all the even binomial coefficients and each is equal to $\frac{2^n}{2} = 2^{n-1}$.

$$ {}^nC_0 + {}^nC_2 + {}^nC_4 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$$