Chapter 7

INTEGRALS

7.1 Overview

7.1.1 Let $\frac{d}{dx}$ F $(x) = f(x)$. Then, we write $\int f(x)\,dx$ = F (x) + C. These integrals are called indefinite integrals or general integrals, C is called a constant of integration. All these integrals differ by a constant.

7.1.2 If two functions differ by a constant, they have the same derivative.

7.1.3 Geometrically, the statement $\int f(x)\,dx$ = F (x) + C = y (say) represents a family of curves. The different values of C correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. Further, the tangents to the curves at the points of intersection of a line $x = a$ with the curves are parallel.

7.1.4 ***Some properties of indefinite integrals***

(i) The process of differentiation and integration are inverse of each other, i.e., $\frac{d}{dx}\int f(x)\,dx = f(x)$ and $\int f'(x)\,dx = f(x) + C$, where C is any arbitrary constant.

(ii) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent. So if f and g are two functions such that $\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\int g(x)\,dx$, then $\int f(x)\,dx$ and $\int g(x)\,dx$ are equivalent.

(iii) The integral of the sum of two functions equals the sum of the integrals of the functions i.e., $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$.

(iv) A constant factor may be written either before or after the integral sign, i.e., $\int a\, f(x)\, dx = a \int f(x)\, dx$, where '$a$' is a constant.

(v) Properties (iii) and (iv) can be generalised to a finite number of functions $f_1, f_2, ..., f_n$ and the real numbers, $k_1, k_2, ..., k_n$ giving

$$\int (k_1 f_1(x) + k_2\, f_2(x) + ... + k_n f_n(x))\, dx = k_1 \int f_1(x) dx + k_2 \int f_2(x) dx + ... + k_n \int f_n(x) dx$$

7.1.5 *Methods of integration*

There are some methods or techniques for finding the integral where we can not directly select the antiderivative of function f by reducing them into standard forms. Some of these methods are based on

1. Integration by substitution
2. Integration using partial fractions
3. Integration by parts.

7.1.6 *Definite integral*

The definite integral is denoted by $\int_a^b f(x)\, dx$, where a is the lower limit of the integral and b is the upper limit of the integral. The definite integral is evaluated in the following two ways:

(i) The definite integral as the limit of the sum

(ii) $\int_a^b f(x)\, dx = F(b) - F(a)$, if F is an antiderivative of $f(x)$.

7.1.7 *The definite integral as the limit of the sum*

The definite integral $\int_a^b f(x)\, dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis and given by

$$\int_a^b f(x)\, dx = (b - a) \lim_{n \to \infty} \frac{1}{n} \left[f(a) + f(a+h) + ... f(a + (n-1)h) \right]$$

or

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + ... + f(a+(n-1)h)\right],$$

where $h = \dfrac{b-a}{n} \to 0$ as $n \to \infty$.

7.1.8 *Fundamental Theorem of Calculus*

(i) *Area function* : The function A (x) denotes the area function and is given by A (x) = $\int_a^x f(x)\,dx$.

(ii) *First Fundamental Theorem of integral Calculus*

Let f be a continuous function on the closed interval [a, b] and let A (x) be the area function . Then A′ (x) = f (x) for all $x \in$ [a, b] .

(iii) *Second Fundamental Theorem of Integral Calculus*

Let f be continuous function defined on the closed interval [a, b] and F be an antiderivative of f.

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a).$$

7.1.9 *Some properties of Definite Integrals*

P_0 : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P_1 : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$, in particular, $\int_a^a f(x)\,dx = 0$

P_2 : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$$P_3 : \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$P_4 : \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$P_5 : \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$P_6 : \int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{if } f(2a-x) = f(x), \\ 0, \text{ if } f(2a-x) = -f(x). \end{cases}$$

P_7 : (i) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function i.e., $f(-x) = f(x)$

(ii) $\int_{-a}^{a} f(x)\,dx = 0$, if f is an odd function i.e., $f(-x) = -f(x)$

7.2 Solved Examples

Short Answer (S.A.)

Example 1 Integrate $\left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right)$ w.r.t. x

Solution $\int\left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right)dx$

$$= \int 2a(x)^{\frac{-1}{2}}\,dx - \int bx^{-2}\,dx + \int 3c\,x^{\frac{2}{3}}\,dx$$

$$= 4a\sqrt{x} + \frac{b}{x} + \frac{9cx^{\frac{5}{3}}}{5} + C\,.$$

Example 2 Evaluate $\int \frac{3ax}{b^2+c^2x^2}dx$

Solution Let $v = b^2 + c^2x^2$, then $dv = 2c^2\, xdx$

Therefore, $\int \frac{3ax}{b^2+c^2x^2}dx = \frac{3a}{2c^2}\int \frac{dv}{v}$

$$= \frac{3a}{2c^2}\log\left|b^2+c^2x^2\right| + \mathrm{C}.$$

Example 3 Verify the following using the concept of integration as an antiderivative.

$$\int \frac{x^3 dx}{x+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \log\left|x+1\right| + \mathrm{C}$$

Solution $\frac{d}{dx}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \log\left|x+1\right| + \mathrm{C}\right)$

$$= 1 - \frac{2x}{2} + \frac{3x^2}{3} - \frac{1}{x+1}$$

$$= 1 - x + x^2 - \frac{1}{x+1} = \frac{x^3}{x+1}.$$

Thus $\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \log\left|x+1\right| + \mathrm{C}\right) = \int \frac{x^3}{x+1}dx$

Example 4 Evaluate $\int \sqrt{\frac{1+x}{1-x}}\, dx,\ x \neq 1.$

Solution Let $\mathrm{I} = \int \sqrt{\frac{1+x}{1-x}}\, dx = \int \frac{1}{\sqrt{1-x^2}}dx + \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1}x + \mathrm{I}_1,$

where $I_1 = \frac{x\,dx}{\sqrt{1-x^2}}$.

Put $1 - x^2 = t^2 \Rightarrow -2x\,dx = 2t\,dt$. Therefore

$$I_1 = - \; dt = -t + C = -\sqrt{1-x^2} + C$$

Hence $\quad I = \sin^{-1}x - \sqrt{1-x^2} + C$.

Example 5 Evaluate $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}, \beta > \alpha$

Solution Put $x - \alpha = t^2$. Then $\beta - x = \beta - \left(t^2 + \alpha\right) = \beta - t^2 - \alpha = -t^2 - \alpha + \beta$ and $dx = 2t\,dt$. Now

$$I = \int \frac{2t\,dt}{\sqrt{t^2\left(\beta - \alpha - t^2\right)}} = \int \frac{2\,dt}{\sqrt{\left(\beta - \alpha - t^2\right)}}$$

$$= 2 \frac{dt}{\sqrt{k^2 - t^2}}, \text{ where } k^2 = \beta - \alpha$$

$$= 2\sin^{-1}\frac{t}{k} + C = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + C.$$

Example 6 Evaluate $\int \tan^8 x \sec^4 x\,dx$

Solution $I = \int \tan^8 x \sec^4 x\,dx$

$$= \int \tan^8 x\left(\sec^2 x\right) \sec^2 x\,dx$$

$$= \int \tan^8 x\left(\tan^2 x + 1\right) \sec^2 x\,dx$$

$$= \int \tan^{10} x \sec^2 x\, dx + \int \tan^8 x \sec^2 x\, dx$$

$$= \frac{\tan^{11} x}{11} + \frac{\tan^9 x}{9} + C\,.$$

Example 7 Find $\int \frac{x^3}{x^4 + 3x^2 + 2} dx$

Solution Put $x^2 = t$. Then $2x\, dx = dt$.

Now $$I = \int \frac{x^3 dx}{x^4 + 3x^2 + 2} = \frac{1}{2} \int \frac{t\, dt}{t^2 + 3t + 2}$$

Consider $$\frac{t}{t^2 + 3t + 2} = \frac{A}{t+1} + \frac{B}{t+2}$$

Comparing coefficient, we get A = –1, B = 2.

Then $$I = \frac{1}{2}\left[2\int \frac{dt}{t+2} - \int \frac{dt}{t+1}\right]$$

$$= \frac{1}{2}\left[2\log|t+2| - \log|t+1|\right]$$

$$= \log\left|\frac{x^2 + 2}{\sqrt{x^2 + 1}}\right| + C$$

Example 8 Find $\int \frac{dx}{2\sin^2 x + 5\cos^2 x}$

Solution Dividing numerator and denominator by $\cos^2 x$, we have

$$I = \int \frac{\sec^2 x\, dx}{2\tan^2 x + 5}$$

Put $\tan x = t$ so that $\sec^2 x\, dx = dt$. Then

$$I = \int \frac{dt}{2t^2 + 5} = \frac{1}{2}\int \frac{dt}{t^2 + \left(\sqrt{\frac{5}{2}}\right)^2}$$

$$= \frac{1}{2}\frac{\sqrt{2}}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{2}t}{\sqrt{5}}\right) + C$$

$$= \frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{\sqrt{2}\tan x}{\sqrt{5}}\right) + C.$$

Example 9 Evaluate $\int_{-1}^{2}(7x-5)\,dx$ as a limit of sums.

Solution Here $a = -1$, $b = 2$, and $h = \frac{2+1}{n}$, i.e, $nh = 3$ and $f(x) = 7x - 5$.

Now, we have

$$\int_{-1}^{2}(7x-5)\,dx = \lim_{h\to 0} h\left[f(-1) + f(-1+h) + f(-1+2h) + ... + f(-1+(n-1)h)\right]$$

Note that

$f(-1) = -7 - 5 = -12$

$f(-1 + h) = -7 + 7h - 5 = -12 + 7h$

$f(-1 + (n-1)h) = 7(n-1)h - 12.$

Therefore,

$$\int_{-1}^{2}(7x-5)\,dx = \lim_{h\to 0} h\left[(-12) + (7h-12) + (14h-12) + ... + (7(n-1)h - 12)\right].$$

$$= \lim_{h\to 0} h\left[7h\left[1 + 2 + ... + (n-1)\right] - 12n\right]$$

$$= \lim_{h\to 0} h\left[7h\frac{(n-1)n}{2} - .12n\right] = \lim_{h\to 0}\left[\frac{7}{2}(nh)(nh-h) - 12nh\right]$$

$$= \frac{7}{2}(3)(3-0) - 12\times 3 = \frac{7\times 9}{2} - 36 = \frac{-9}{2}.$$

Example 10 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\cot^7 x + \tan^7 x} dx$

Solution We have

$$\text{I} = \int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\cot^7 x + \tan^7 x} dx \qquad \text{...(1)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^7\left(\frac{\pi}{2} - x\right)}{\cot^7\left(\frac{\pi}{2} - x\right) + \tan^7\left(\frac{\pi}{2} - x\right)} dx \qquad \text{by } (\text{P}_4)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cot^7(x)\, dx}{\cot^7 x\, dx + \tan^7 x} \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$2\text{I} = \int_0^{\frac{\pi}{2}} \left(\frac{\tan^7 x + \cot^7 x}{\tan^7 x + \cot^7 x}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} dx \text{ which gives } \text{I} = \frac{\pi}{4}.$$

Example 11 Find $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx$

Solution We have

$$I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx \qquad ...(1)$$

$$= \int_2^8 \frac{\sqrt{10-(10-x)}}{\sqrt{10-x}+\sqrt{10-(10-x)}}dx \qquad \text{by } (P_3)$$

$$\Rightarrow \qquad I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx \qquad (2)$$

Adding (1) and (2), we get

$$2I = \int_2^8 1dx = 8-2=6$$

Hence $\quad I = 3$

Example 12 Find $\int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x}\, dx$

Solution We have

$$I = \int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{(\sin x+\cos x)^2}\, dx$$

$$= \int_0^{\frac{\pi}{4}} (\sin x+\cos x)\, dx$$

$$= \left(-\cos x + \sin x\right)_0^{\frac{\pi}{4}}$$

$$I = 1.$$

Example 13 Find $\int x^2 \tan^{-1} x \, dx$.

Solution $I = \int x^2 \tan^{-1} x \, dx$

$$= \tan^{-1} x \int x^2 dx - \int \frac{1}{1+x^2} \cdot \frac{x^3}{3} dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \log\left|1 + x^2\right| + C.$$

Example 14 Find $\int \sqrt{10 - 4x + 4x^2} \, dx$

Solution We have

$$I = \int \sqrt{10 - 4x + 4x^2} \, dx = \int \sqrt{(2x-1)^2 + (3)^2} \, dx$$

Put $t = 2x - 1$, then $dt = 2dx$.

Therefore, $I = \frac{1}{2} \int \sqrt{t^2 + (3)^2} \, dt$

$$= \frac{1}{2} t \frac{\sqrt{t^2+9}}{2} + \frac{9}{4} \log\left|t + \sqrt{t^2+9}\right| + C$$

$$= \frac{1}{4}(2x-1)\sqrt{(2x-1)^2+9} + \frac{9}{4} \log\left|(2x-1) + \sqrt{(2x-1)^2+9}\right| + C.$$

Long Answer (L.A.)

Example 15 Evaluate $\int \frac{x^2\,dx}{x^4 + x^2 - 2}$.

Solution Let $x^2 = t$. Then

$$\frac{x^2}{x^4 + x^2 - 2} = \frac{t}{t^2 + t - 2} = \frac{t}{(t+2)\,(t-1)} = \frac{A}{t+2} + \frac{B}{t-1}$$

So $\quad t = A\,(t - 1) + B\,(t + 2)$

Comparing coefficients, we get $A = \frac{2}{3},\ B = \frac{1}{3}$.

So $$\frac{x^2}{x^4 + x^2 - 2} = \frac{2}{3}\frac{1}{x^2+2} + \frac{1}{3}\frac{1}{x^2-1}$$

Therefore,

$$\int \frac{x^2}{x^4 + x^2 - 2}\,dx = \frac{2}{3}\int \frac{1}{x^2+2}\,dx + \frac{1}{3}\int \frac{dx}{x^2-1}$$

$$= \frac{2}{3}\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + C$$

Example16 Evaluate $\int \frac{x^3 + x}{x^4 \quad 9}\,dx$

Solution We have

$$I = \int \frac{x^3 + x}{x^4 \quad 9}\,dx \quad = \int \frac{x^3}{x^4 \quad 9}\,dx + \frac{x\,dx}{x^4 \quad 9} \quad = I_1 + I_2\,.$$

Now $\quad I_1 = \int \frac{x^3}{x^4 - 9}$

Put $\quad t = x^4 - 9$ so that $4x^3\, dx = dt$. Therefore

$$I_1 = \frac{1}{4}\int \frac{dt}{t} = \frac{1}{4}\log|t| + C_1 = \frac{1}{4}\log\left|x^4 - 9\right| + C_1$$

Again, $\quad I_2 = \int \frac{x\,dx}{x^4 - 9}$.

Put $\quad x^2 = u$ so that $2x\, dx = du$. Then

$$I_2 = \frac{1}{2}\int \frac{du}{u^2 - (3)^2} = \frac{1}{2 \times 6}\log\left|\frac{u-3}{u+3}\right| + C_2$$

$$= \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C_2.$$

Thus $\quad I = I_1 + I_2$

$$= \frac{1}{4}\log\left|x^4 - 9\right| + \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C.$$

Example 17 Show that $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

Solution We have

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx \qquad \text{(by P4)}$$

$\Rightarrow$ $$\text{I} = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x+\cos x}dx$$

Thus, we get $$2\text{I} = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dx}{\cos\left(x-\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \sec\left(x-\frac{\pi}{4}\right)dx = \frac{1}{\sqrt{2}}\left[\log\left(\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)-\log\sec\left(-\frac{\pi}{4}\right)+\tan\left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sqrt{2}+1\right)-\log\left(\sqrt{2}-1\right)\right] \qquad = \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

$$= \frac{1}{\sqrt{2}}\log\left(\frac{(\sqrt{2}+1)^2}{1}\right) \qquad = \frac{2}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

Hence $$\text{I} = \frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right).$$

Example 18 Find $\int_0^1 x\left(\tan^{-1} x\right)^2 dx$

Solution $\quad \mathrm{I} = \int_0^1 x\left(\tan^{-1} x\right)^2 dx$.

Integrating by parts, we have

$$\mathrm{I} = \frac{x^2}{2}\left[\left(\tan^{-1} x\right)^2\right]_0^1 - \frac{1}{2}\int_0^1 x^2 . 2\frac{\tan^{-1} x}{1+x^2}\,dx$$

$$= \frac{\pi^2}{32} - \int_0^1 \frac{x^2}{1+x^2} . \tan^{-1} x\,dx$$

$$= \frac{\pi^2}{32} - \mathrm{I}_1 \text{, where } \mathrm{I}_1 = \int_0^1 \frac{x^2}{1+x^2} \tan^{-1} x dx$$

Now $\quad \mathrm{I}_1 = \int_0^1 \frac{x^2+1-1}{1+x^2} \tan^{-1} x\, dx$

$$= \int_0^1 \tan^{-1} x\,dx - \int_0^1 \frac{1}{1+x^2} \tan^{-1} x\,dx$$

$$= \mathrm{I}_2 - \frac{1}{2}\left(\left(\tan^{-1} x\right)^2\right)_0^1 \quad = \mathrm{I}_2 - \frac{\pi^2}{32}$$

Here $\quad \mathrm{I}_2 = \int_0^1 \tan^{-1} x\,dx = \left(x \tan^{-1} x\right)_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$

$$= \frac{\pi}{4} - \frac{1}{2}\left(\log\left|1+x^2\right|\right)_0^1 \quad = \frac{\pi}{4} - \frac{1}{2}\log 2 .$$

Thus $\quad \mathrm{I}_1 = \frac{\pi}{4} - \frac{1}{2}\log 2 - \frac{\pi^2}{32}$

Therefore, $$I = \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{1}{2}\log 2 + \frac{\pi^2}{32} = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log 2$$

$$= \frac{\pi^2 - 4\pi}{16} + \log\sqrt{2}.$$

Example 19 Evaluate $\int_{-1}^{2} f(x)\, dx$, where $f(x) = |x + 1| + |x| + |x - 1|$.

Solution We can redefine f as $f(x) = \begin{cases} 2 - x, & \text{if } -1 < x \le 0 \\ x + 2, & \text{if } 0 < x \le 1 \\ 3x, & \text{if } 1 < x \le 2 \end{cases}$

Therefore, $$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{0} (2 - x)\,dx + \int_{0}^{1} (x + 2)\,dx + \int_{1}^{2} 3x\,dx \qquad \text{(by } P_2\text{)}$$

$$= \left(2x - \frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2} + 2x\right)_{0}^{1} + \left(\frac{3x^2}{2}\right)_{1}^{2}$$

$$= 0 - \left(-2 - \frac{1}{2}\right) + \left(\frac{1}{2} + 2\right) + 3\left(\frac{4}{2} - \frac{1}{2}\right) = \frac{5}{2} + \frac{5}{2} + \frac{9}{2} = \frac{19}{2}.$$

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples from 20 to 30.

Example 20 $\int e^x (\cos x - \sin x)\, dx$ is equal to

(A) $e^x \cos x + C$ (B) $e^x \sin x + C$

(C) $-e^x \cos x + C$ (D) $-e^x \sin x + C$

Solution (A) is the correct answer since $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$. Here $f(x) = \cos x$, $f'(x) = -\sin x$.

Example 21 $\int \frac{dx}{\sin^2 x \cos^2 x}$ is equal to

(A) $\tan x + \cot x + C$
(B) $(\tan x + \cot x)^2 + C$
(C) $\tan x - \cot x + C$
(D) $(\tan x - \cot x)^2 + C$

Solution (C) is the correct answer, since

$$I = \int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)\, dx}{\sin^2 x \cos^2 x}$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx = \tan x - \cot x + C$$

Example 22 If $\int \frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}}\, dx = ax + b \log |4e^x + 5e^{-x}| + C$, then

(A) $a = \frac{-1}{8}, b = \frac{7}{8}$
(B) $a = \frac{1}{8}, b = \frac{7}{8}$
(C) $a = \frac{-1}{8}, b = \frac{-7}{8}$
(D) $a = \frac{1}{8}, b = \frac{-7}{8}$

Solution (C) is the correct answer, since differentiating both sides, we have

$$\frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}} = a + b\,\frac{(4e^x - 5e^{-x})}{4e^x + 5e^{-x}},$$

giving $3e^x - 5e^{-x} = a(4e^x + 5e^{-x}) + b(4e^x - 5e^{-x})$. Comparing coefficients on both sides, we get $3 = 4a + 4b$ and $-5 = 5a - 5b$. This verifies $a = \frac{-1}{8}, b = \frac{7}{8}$.

Example 23 $\int_{a+c}^{b+c} f(x)\,dx$ is equal to

(A) $\int_a^b f(x-c)\,dx$ (B) $\int_a^b f(x+c)\,dx$

(C) $\int_a^b f(x)\,dx$ (D) $\int_{a-c}^{b-c} f(x)\,dx$

Solution (B) is the correct answer, since by putting $x = t + c$, we get

$$I = \int_a^b f(c+t)\,dt = \int_a^b f(x+c)\,dx.$$

Example 24 If f and g are continuous functions in [0, 1] satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = a$, then $\int_0^a f(x).g(x)\,dx$ is equal to

(A) $\frac{a}{2}$ (B) $\frac{a}{2}\int_0^a f(x)\,dx$

(C) $\int_0^a f(x)\,dx$ (D) $a\int_0^a f(x)\,dx$

Solution B is the correct answer. Since $I = \int_0^a f(x).g(x)\,dx$

$$= \int_0^a f(a-x)\,g(a-x)\,dx = \int_0^a f(x)\,(a-g(x))\,dx$$

$$= a\int_0^a f(x)\,dx - \int_0^a f(x).g(x)\,dx = a\int_0^a f(x)\,dx - I$$

or $\quad I = \frac{a}{2}\int_0^a f(x)\,dx$.

Example 25 If $x = \int_0^y \frac{dt}{\sqrt{1+9t^2}}$ and $\frac{d^2y}{dx^2} = ay$, then a is equal to

(A) 3 (B) 6 (C) 9 (D) 1

Solution (C) is the correct answer, since $x = \int_0^y \frac{dt}{\sqrt{1+9t^2}} \Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{1+9y^2}}$

which gives $\frac{d^2y}{dx^2} = \frac{18y}{2\sqrt{1+9y^2}} \cdot \frac{dy}{dx} = 9y$.

Example 26 $\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx$ is equal to

(A) log 2 (B) 2 log 2 (C) $\frac{1}{2}\log 2$ (D) 4 log 2

Solution (B) is the correct answer, since I = $\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx$

$$= \int_{-1}^{1} \frac{x^3}{x^2+2|x|+1} + \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1}\,dx \quad = \quad 0 + 2\int_0^1 \frac{|x|+1}{(|x|+1)^2}\,dx$$

[odd function + even function]

$$= 2\int_0^1 \frac{x+1}{(x+1)^2}\,dx = 2\int_0^1 \frac{1}{x+1}\,dx \qquad = 2\left|\log|x+1|\right|_0^1 = 2\log 2.$$

Example 27 If $\int_0^1 \frac{e^t}{1+t}dt = a$, then $\int_0^1 \frac{e^t}{(1+t)^2}dt$ is equal to

(A) $a - 1 + \frac{e}{2}$ (B) $a + 1 - \frac{e}{2}$ (C) $a - 1 - \frac{e}{2}$ (D) $a + 1 + \frac{e}{2}$

Solution (B) is the correct answer, since I = $\int_0^1 \frac{e^t}{1+t}dt$

$$= \left|\frac{1}{1+t}e^t\right|_0^1 + \int_0^1 \frac{e^t}{(1+t)^2}dt = a \text{ (given)}$$

Therefore, $\int_0^1 \frac{e^t}{(1+t)^2} = a - \frac{e}{2} + 1.$

Example 28 $\int_{-2}^{2} |x\cos \pi x|\,dx$ is equal to

(A) $\frac{8}{\pi}$ (B) $\frac{4}{\pi}$ (C) $\frac{2}{\pi}$ (D) $\frac{1}{\pi}$

Solution (A) is the correct answer, since I = $\int_{-2}^{2} |x\cos \pi x|\,dx = 2\int_0^2 |x\cos \pi x|\,dx$

$$= 2\left\{\int_0^{\frac{1}{2}} |x\cos \pi x|\,dx + \int_{\frac{1}{2}}^{\frac{3}{2}} |x\cos \pi x|\,dx + \int_{\frac{3}{2}}^{2} |x\cos \pi x|\,dx\right\} = \frac{8}{\pi}.$$

Fill in the blanks in each of the Examples 29 to 32.

Example 29 $\int \frac{\sin^6 x}{\cos^8 x}dx =$ _______.

Solution $\frac{\tan^7 x}{7} + C$

Example 30 $\int_{-a}^{a} f(x)\,dx = 0$ if f is an ________ function.

Solution Odd.

Example 31 $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$, if $f(2a - x) =$ ________.

Solution $f(x)$.

Example 32 $\int_0^{\frac{\pi}{2}} \frac{\sin^n x\,dx}{\sin^n x + \cos^n x} =$ ________.

Solution $\frac{\pi}{4}$.

7.3 EXERCISE

Short Answer (S.A.)

Verify the following :

1. $\int \frac{2x-1}{2x+3}dx = x - \log|(2x+3)^2| + C$

2. $\int \frac{2x+3}{x^2+3x}dx = \log|x^2+3x| + C$

Evaluate the following:

3. $\int \frac{(x^2+2)\,dx}{x+1}$

4. $\int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}dx$