Lecture-2

Eg. In how many ways you can choose 5 numbers a, b, c, d, e such that each one is greater than 0 & $a+b+c+d+e=20$

Solⁿ: No. of possible ways $= {}^{20-1}C_{5-1} = {}^{19}C_4$

If $a, b, c, d, e \geq 0$,

then no. of possible ways $= {}^{20+5-1}C_{5-1} = {}^{24}C_4$

Eg. In how many you can choose 5 numbers n_1, n_2, n_3, n_4 & n_5 $\ni n_i > 0 \ \forall \ i = 1, 5$ & $n_1 < n_2 < n_3 < n_4 < n_5$

and $\sum_{i=1}^{5} n_i = 20$

Solⁿ

This means that all numbers must be positive and their sum must be 20.
But they must all be in ascending order.

eg. (1, 2, 3, 4, 10) is a solution

but (1, 2, 4, 4, 9) is not a solution because 4 ~~is rep~~ 4 is repeated.

Let the first number be n_1 & second number n_2 and so on.

The smallest value for $n_1 = 1$ and for $n_2 = 2$ and so on -- for $n_5 = 5$. because the numbers must be in increasing order.

Let us define 5 new variables
x_1, x_2, x_3, x_4, x_5 such that

$x_1 = n_1 - 1$
$x_2 = n_2 - 2$ So, Each $x_i \leq 0$
$x_3 = n_3 - 3$
$x_4 = n_4 - 4$ and $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$
$x_5 = n_5 - 5$

Let us add all x_i

$$x_1 + x_2 + x_3 + x_4 + x_5 = n_1 + n_2 + n_3 + n_4 + n_5 - 15$$
$$= 20 - 15$$
$$= 5$$

So, the problem is now to choose x_1, x_2, x_3, x_4, x_5 such that $x_i \geq 0$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 5$

For every such possibilities of x_i there will be a possibility of n_i

for eg. If $x_1 = 0$, then $n_1 = x_1 + 1 = 1$
$x_2 = 0$ $n_2 = x_2 + 2 = 2$
$x_3 = 0$ $n_3 = x_3 + 3 = 3$
$x_4 = 0$ $n_4 = x_4 + 4 = 4$
$x_5 = \cancel{0}\ 5$ $n_5 = x_5 + 5 = \cancel{5}\ 10$

But, here we cannot use ${}^{n+k-1}C_{k-1}$ because, in this case, the numbers must be in ascending order.

So, we must choose the possibilities manually.

x_1	x_2	x_3	x_4	x_5	n_1	n_2	n_3	n_4	n_5
0	0	0	0	5	1	2	3	4	10
0	0	0	1	4	1	2	3	5	9
0	0	0	2	3	1	2	3	6	8
0	0	1	1	3	1	2	4	5	8
0	0	1	2	2	1	2	4	6	7
0	1	1	1	2	1	3	4	5	7
1	1	1	1	1	2	3	4	5	6

Number of possible solutions is 7.

Probability:

$$\text{Probability of an event} = \frac{\text{no. of elements in event}}{\text{no. of elements in } \Omega}$$

An event is a subset of the sample space Ω.

Subset of cardinality 1 is called elementary event

If it has more than one elements, it is called a compound event.

Total no. of possible subsets $= 2^n$
$n \rightarrow$ cardinality of Ω

Example: Suppose Ω is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

How many compound events are possible?

Solⁿ: $|\Omega| = 9$

There are 2^9 subsets

of which 9 are elementary

and 1 is ϕ (empty)

So, number of compound events $= 2^9 - (9+1)$

$= 502$

Disjoint events:

Two events E_1 & E_2 are disjoint if

$$E_1 \cap E_2 = \phi$$

Eg. Throwing a dies

E_1 : getting a number ≤ 3] → disjoint
E_2 : getting a number ≥ 4

Mutually exclusive events:

A sequence of events $E_1, E_2, \ldots, E_K$ is said to be mutually exclusive

if $E_1 \cap E_2 \cap \cdots \cap E_K = \phi$

Independent events:

Two events are independent if

$$P(A \cap B) = P(A) \times P(B)$$

Probability of an event:

Probability is a mapping from the power set of Ω to $[0,1]$

This means that, power set contains all possible events of an experiment and every event is ~~related~~ ~~connected~~ to a number between 0 and 1.

If event $A \subseteq \Omega$, then $P(A) = p \ni 0 \leq p \leq 1$

And i) $P(A) \geq 0 \quad \forall A \subseteq \Omega$

This means probability of any event is greater than or equal to zero.

ii) $P(\Omega) = 1$

Probability of sample space is 1.

iii) If $A_1, A_2, \ldots, A_K$ are mutually exclusive then

$$P(A_1 \cup A_2 \cdots \cup A_K) = \sum_{i=1}^{K} P(A_i)$$

- Given a set ~~of Ω~~ $A \subseteq \Omega$.

$P(A)$ is computed as $\dfrac{\text{no. of elements in } A}{|\Omega|}$

when all the outcomes are equally likely.

- $P(A^c) = 1 - P(A)$

Eg. Suppose probability of getting a head is p.

Then probability of getting a tail is $1-p$

Suppose we have to find probability of getting 2H's in 3 tosses.

We can get HHT, THH or HTH

So, $P(E) = P(HHT) + P(HTH) + P(THH)$

$$\begin{matrix} H & H & T \\ \downarrow & \downarrow & \downarrow \\ p & p & (1-p) \end{matrix}$$

$= p^2(1-p) + p^2(1-p) + p^2(1-p)$

$= 3p^2(1-p)$

For a normal coin, $p = \frac{1}{2}$

So, $P(E) = \frac{3}{8}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Eg. Suppose E is a random experiment.

Let A and B be two events $\ni$

$0 < P(A) < 1$ and $0 < P(B) < 1$

Then which of the following statements are true.

A) A and A^c are mutually exclusive

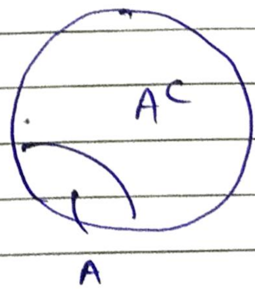
B) A and A^c are independent

C) A and B are independent
$\Rightarrow$ A and B^c are independent

D) A and B are independent
$\Rightarrow$ A^c and B^c are independent

Solution:

A) It is obvious that A and A^c are mutually exclusive.
They have nothing in common.



B) A and B are independent if

$$P(A \cap B) = P(A) \times P(B)$$

$P(A \cap A^c) = P(\phi) = 0$

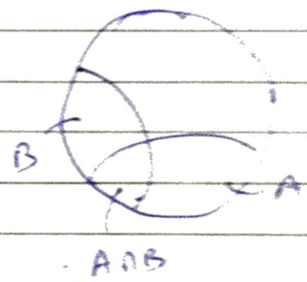
$P(A) \times P(A^c) = p(1-p) \neq 0$

So, $P(A \cap A^c) \neq P(A) \times P(A^c)$

B) is wrong.

c) Given A & B are independent.

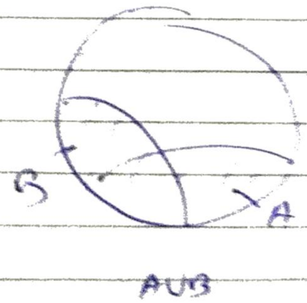
So, $P(A \cap B) = P(A) \times P(B)$



$$
\begin{aligned}
P(A \cap B^c) &= P(A) - P(A \cap B) \\
&= P(A) - P(A)\,P(B) \\
&= P(A)\,(1 - P(B)) \\
&= P(A)\,P(B^c)
\end{aligned}
$$

$\Rightarrow$ A and B^c are independent ✓

d) $P(A^c \cap B^c) = P(\Omega) - P(A \cup B)$



$$
\begin{aligned}
&= 1 - (P(A) + P(B) - P(A \cap B)) \\
&= (1 - P(A)) - P(B)(1 - P(A)) \\
&= P(A^c) - P(B)\,P(A^c) \\
&= P(A^c)\,(1 - P(B)) \\
&= P(A^c)\,P(B^c)
\end{aligned}
$$

$\Rightarrow$ A^c and B^c are independent ✓

$\Rightarrow$ A, C, D are correct