Chapter 2

RELATIONS AND FUNCTIONS

2.1 Overview

This chapter deals with linking pair of elements from two sets and then introduce relations between the two elements in the pair. Practically in every day of our lives, we pair the members of two sets of numbers. For example, each hour of the day is paired with the local temperature reading by T.V. Station's weatherman, a teacher often pairs each set of score with the number of students receiving that score to see more clearly how well the class has understood the lesson. Finally, we shall learn about special relations called functions.

2.1.1 *Cartesian products of sets*

Definition : Given two non-empty sets A and B, the set of all ordered pairs (x, y), where $x \in$ A and $y \in$ B is called Cartesian product of A and B; symbolically, we write

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

If $\quad A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then

$$A \times B = \{(1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5)\}$$

and $\quad B \times A = \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}$

(i) Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal, i.e. $(x, y) = (u, v)$ if and only if $x = u$, $y = v$.

(ii) If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = p \times q$.

(iii) $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$. Here (a, b, c) is called an ordered triplet.

2.1.2 *Relations* A Relation R from a non-empty set A to a non empty set B is a subset of the Cartesian product set A × B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A × B.

The set of all first elements in a relation R, is called the domain of the relation R, and the set of all second elements called images, is called the range of R.

For example, the set $R = \{(1, 2), (-2, 3), (\frac{1}{2}, 3)\}$ is a relation; the domain of $R = \{1, -2, \frac{1}{2}\}$ and the range of $R = \{2, 3\}$.

(i) A relation may be represented either by the Roster form or by the set builder form, or by an arrow diagram which is a visual representation of a relation.

(ii) If n (A) = p, n (B) = q; then the n (A × B) = pq and the total number of possible relations from the set A to set B = 2^{pq}.

2.1.3 *Functions* A relation f from a set A to a set B is said to be **function** if every element of set A has one and only one image in set B.

In other words, a function f is a relation such that no two pairs in the relation has the same first element.

The notation $f: X \to Y$ means that f is a function from X to Y. X is called the **domain** of f and Y is called the **co-domain** of f. Given an element $x \in$ X, there is a unique element y in Y that is related to x. The unique element y to which f relates x is denoted by $f(x)$ and is called f of x, or the **value of f at x**, or the *image of x under f.*

The set of all values of $f(x)$ taken together is called the **range of f** or image of X under f. Symbolically.

$$\text{range of } f = \{ y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}$$

Definition : A function which has either **R** or one of its subsets as its range, is called a real valued function. Further, if its domain is also either **R** or a subset of **R**, it is called a real function.

2.1.4 *Some specific types of functions*

(i) **Identity function:**

The function $f: \mathbf{R} \to \mathbf{R}$ defined by $y = f(x) = x$ for each $x \in \mathbf{R}$ is called the **identity function.** Domain of f = **R**

Range of f = **R**

(ii) **Constant function:** The function $f: \mathbf{R} \to \mathbf{R}$ defined by $y = f(x) = C$, $x \in \mathbf{R}$, where C is a constant $\in \mathbf{R}$, is a **constant function**.

Domain of f = **R**

Range of f = {C}

(iii) **Polynomial function:** A real valued function $f: \mathbf{R} \to \mathbf{R}$ defined by $y = f(x) = a_0 + a_1 x + ... + a_n x^n$, where $n \in \mathbf{N}$, and $a_0, a_1, a_2 ... a_n \in \mathbf{R}$, for each $x \in \mathbf{R}$, is called Polynomial functions.

(iv) **Rational function:** These are the real functions of the type $\dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$. For example $f : \mathbf{R} - \{-2\} \to \mathbf{R}$ defined by $f(x) = \dfrac{x+1}{x+2}$, $\forall\, x \in \mathbf{R} - \{-2\}$ is a rational function.

(v) **The Modulus function:** The real function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = |x| =$

$$\begin{matrix} x, & x \geq 0 \\ -x, & x < 0 \end{matrix}$$

$\forall\, x \in \mathbf{R}$ is called the modulus function.

Domain of $f = \mathbf{R}$

Range of $f = \mathbf{R}^+ \cup \{0\}$

(vi) **Signum function:** The real function
$f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is called the **signum function**. Domain of $f = \mathbf{R}$, Range of $f = \{1, 0, -1\}$

(vii) **Greatest integer function:** The real function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = [x], x \in \mathbf{R}$ assumes the value of the greatest integer less than or equal to x, is called the **greatest integer function**.

Thus $f(x) = [x] = -1$ for $-1 \leq x < 0$
$f(x) = [x] = 0$ for $0 \leq x < 1$
$[x] = 1$ for $1 \leq x < 2$
$[x] = 2$ for $2 \leq x < 3$ and so on

2.1.5 *Algebra of real functions*

(i) ***Addition of two real functions***

Let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be any two real functions, where $X \in \mathbf{R}$.
Then we define $(f + g) : X \to \mathbf{R}$ by $(f + g)(x) = f(x) + g(x)$, for all $x \in X$.

(ii) ***Subtraction of a real function from another***

Let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be any two real functions, where $X \subseteq \mathbf{R}$.

Then, we define $(f - g) : X \to \mathbf{R}$ by $(f - g)(x) = f(x) - g(x)$, for all $x \in X$.

(iii) ***Multiplication by a Scalar***

Let $f : X \to \mathbf{R}$ be a real function and α be any scalar belonging to $\mathbf{R}$. Then the product αf is function from X to $\mathbf{R}$ defined by $(\alpha f)(x) = \alpha f(x)$, $x \in X$.

(iv) ***Multiplication of two real functions***

Let $f: \mathrm{X} \to \mathbf{R}$ and $g : x \to \mathbf{R}$ be any two real functions, where $\mathrm{X} \subseteq \mathbf{R}$. Then product of these two functions i.e. $f\ g : \mathrm{X} \to \mathbf{R}$ is defined by $(f\,g)\,(x) = f(x)\ \ g\,(x)\ \forall\ x \in \mathrm{X}$.

(v) ***Quotient of two real function***

Let f and g be two real functions defined from $\mathrm{X} \to \mathbf{R}$. The quotient of f by g denoted by $\dfrac{f}{g}$ is a function defined from $\mathrm{X} \to \mathbf{R}$ as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ \text{provided } g\,(x) \neq 0,\ x \in \mathrm{X}.$$

Note Domain of sum function $f + g$, difference function $f - g$ and product function fg.

$$= \{x : x \in \mathrm{D}_f \cap \mathrm{D}_g\}$$

where D_f = Domain of function f

D_g = Domain of function g

Domain of quotient function $\dfrac{f}{g}$

$$= \{x : x \in \mathrm{D}_f \cap \mathrm{D}_g \text{ and } g\,(x) \neq 0\}$$