

EXERCISE 3.3

Prove that:

1. $\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} = -\frac{1}{2}$
2. $2\sin^2 \frac{\pi}{6} + \operatorname{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} = \frac{3}{2}$
3. $\cot^2 \frac{\pi}{6} + \operatorname{cosec} \frac{5\pi}{6} + 3\tan^2 \frac{\pi}{6} = 6$
4. $2\sin^2 \frac{3\pi}{4} + 2\cos^2 \frac{\pi}{4} + 2\sec^2 \frac{\pi}{3} = 10$
5. Find the value of:
 - (i) $\sin 75^\circ$
 - (ii) $\tan 15^\circ$
6. Prove the following:

$$\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right)\sin\left(\frac{\pi}{4} - y\right) = \sin(x + y)$$
7. $\frac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \left(\frac{1 + \tan x}{1 - \tan x}\right)^2$
8. $\frac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\frac{\pi}{2} + x\right)} = \cot^2 x$
9. $\cos\left(\frac{3\pi}{2} + x\right)\cos(2\pi + x)\left[\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\right] = 1$
10. $\sin(n + 1)x \sin(n + 2)x + \cos(n + 1)x \cos(n + 2)x = \cos x$
11. $\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2}\sin x$
12. $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$
13. $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$
14. $\sin 2x + 2 \sin 4x + \sin 6x = 4 \cos^2 x \sin 4x$
15. $\cot 4x (\sin 5x + \sin 3x) = \cot x (\sin 5x - \sin 3x)$
16. $\frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\frac{\sin 2x}{\cos 10x}$
17. $\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$
18. $\frac{\sin x - \sin y}{\cos x + \cos y} = \tan \frac{x - y}{2}$
19. $\frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$
20. $\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2 \sin x$
21. $\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

$$22. \cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$$

$$23. \tan 4x = \frac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x} \qquad 24. \cos 4x = 1 - 8 \sin^2 x \cos^2 x$$

$$25. \cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1$$

3.5 Trigonometric Equations

Equations involving trigonometric functions of a variable are called *trigonometric equations*. In this Section, we shall find the solutions of such equations. We have already learnt that the values of $\sin x$ and $\cos x$ repeat after an interval of 2π and the values of $\tan x$ repeat after an interval of π . The solutions of a trigonometric equation for which $0 \leq x < 2\pi$ are called *principal solutions*. The expression involving integer ' n ' which gives all solutions of a trigonometric equation is called the *general solution*. We shall use ' \mathbf{Z} ' to denote the set of integers.

The following examples will be helpful in solving trigonometric equations:

Example 18 Find the principal solutions of the equation $\sin x = \frac{\sqrt{3}}{2}$.

Solution We know that, $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin \frac{2\pi}{3} = \sin \left(\pi - \frac{\pi}{3} \right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Therefore, principal solutions are $x = \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

Example 19 Find the principal solutions of the equation $\tan x = -\frac{1}{\sqrt{3}}$.

Solution We know that, $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$. Thus, $\tan \left(\pi - \frac{\pi}{6} \right) = -\tan \frac{\pi}{6} = -\frac{1}{\sqrt{3}}$

and $\tan \left(2\pi - \frac{\pi}{6} \right) = -\tan \frac{\pi}{6} = -\frac{1}{\sqrt{3}}$

Thus $\tan \frac{5\pi}{6} = \tan \frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$.

Therefore, principal solutions are $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$.

We will now find the general solutions of trigonometric equations. We have already

seen that:

$$\sin x = 0 \text{ gives } x = n\pi, \text{ where } n \in \mathbf{Z}$$

$$\cos x = 0 \text{ gives } x = (2n + 1)\frac{\pi}{2}, \text{ where } n \in \mathbf{Z}.$$

We shall now prove the following results:

Theorem 1 For any real numbers x and y ,

$$\sin x = \sin y \text{ implies } x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}$$

Proof If $\sin x = \sin y$, then

$$\sin x - \sin y = 0 \text{ or } 2\cos \frac{x+y}{2} \sin \frac{x-y}{2} = 0$$

which gives $\cos \frac{x+y}{2} = 0$ or $\sin \frac{x-y}{2} = 0$

Therefore $\frac{x+y}{2} = (2n+1)\frac{\pi}{2}$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

i.e. $x = (2n+1)\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

Hence $x = (2n+1)\pi + (-1)^{2n+1}y$ or $x = 2n\pi + (-1)^{2n}y$, where $n \in \mathbf{Z}$.

Combining these two results, we get

$$x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}.$$

Theorem 2 For any real numbers x and y , $\cos x = \cos y$, implies $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Proof If $\cos x = \cos y$, then

$$\cos x - \cos y = 0 \text{ i.e., } -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2} = 0$$

Thus $\sin \frac{x+y}{2} = 0$ or $\sin \frac{x-y}{2} = 0$

Therefore $\frac{x+y}{2} = n\pi$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

i.e. $x = 2n\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

Hence $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Theorem 3 Prove that if x and y are not odd multiple of $\frac{\pi}{2}$, then

$$\tan x = \tan y \text{ implies } x = n\pi + y, \text{ where } n \in \mathbf{Z}$$

Proof If $\tan x = \tan y$, then $\tan x - \tan y = 0$

or
$$\frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y} = 0$$

which gives $\sin(x - y) = 0$ (Why?)


Therefore $x - y = n\pi$, i.e., $x = n\pi + y$, where $n \in \mathbf{Z}$

Example 20 Find the solution of $\sin x = -\frac{\sqrt{3}}{2}$.

Solution We have $\sin x = -\frac{\sqrt{3}}{2} = -\sin \frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin \frac{4\pi}{3}$

Hence $\sin x = \sin \frac{4\pi}{3}$, which gives

$$x = n\pi + (-1)^n \frac{4\pi}{3}, \text{ where } n \in \mathbf{Z}.$$

 **Note** $\frac{4\pi}{3}$ is one such value of x for which $\sin x = -\frac{\sqrt{3}}{2}$. One may take any other value of x for which $\sin x = -\frac{\sqrt{3}}{2}$. The solutions obtained will be the same although these may apparently look different.

Example 21 Solve $\cos x = \frac{1}{2}$.

Solution We have, $\cos x = \frac{1}{2} = \cos \frac{\pi}{3}$

Therefore $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in \mathbf{Z}$.

Example 22 Solve $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$.

Solution We have, $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{2} + x + \frac{\pi}{3}\right)$