Chapter 11

THREE DIMENSIONAL GEOMETRY

11.1 Overview

11.1.1 Direction cosines of a line are the cosines of the angles made by the line with positive directions of the co-ordinate axes.

11.1.2 If l, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$

11.1.3 Direction cosines of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ},$$

where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

11.1.4 Direction ratios of a line are the numbers which are proportional to the direction cosines of the line.

11.1.5 If l, m, n are the direction cosines and a, b, c are the direction ratios of a line,

then $l = \frac{\pm a}{\sqrt{a^2 + b^2 + c^2}}; m = \frac{\pm b}{\sqrt{a^2 + b^2 + c^2}}; n = \frac{\pm c}{\sqrt{a^2 + b^2 + c^2}}$

11.1.6 Skew lines are lines in the space which are neither parallel nor interesecting. They lie in the different planes.

11.1.7 Angle between skew lines is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.

11.1.8 If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines and θ is the acute angle between the two lines, then

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

11.1.9 If a_1, b_1, c_1 and a_2, b_2, c_2 are the directions ratios of two lines and θ is the acute angle between the two lines, then

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,.\sqrt{b_1^2 + b_2^2 + b_3^2}}\right|$$

11.1.10 Vector equation of a line that passes through the given point whose position vector is a and parallel to a given vector b is $r = a + \lambda b$.

11.1.11 Equation of a line through a point (x_1, y_1, z_1) and having directions cosines l, m, n (or, direction ratios a, b and c) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \text{ or } \left(\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}\right).$$

11.1.12 The vector equation of a line that passes through two points whose positions vectors are a and b is $r = a + \lambda(b - a)$.

11.1.13 Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

11.1.14 If θ is the acute angle between the lines $r = a_1 + \lambda b_1$ and $r = a_2 + \lambda b_2$, then θ is given by $\cos\theta = \frac{|b_1 . b_2|}{|b_1||b_2|}$ or $\theta = \cos^{-1} \frac{|b_1 . b_2|}{|b_1||b_2|}$.

11.1.15 If $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_1} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ are equations of two lines, then the acute angle θ between the two lines is given by $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$.

11.1.16 The shortest distance between two skew lines is the length of the line segment perpendicular to both the lines.

11.1.17 The shortest distance between the lines $r = a_1 + \lambda b_1$ and $r = a_2 + \lambda b_2$ is

$$\left|\frac{(b_1 \times b_2).(a_2 - a_1)}{|b_1 \times b_2|}\right|.$$

11.1.18 Shortest distance between the lines: $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

11.1.19 Distance between parallel lines $r=a_1+\mu b$ and $r=a_2+\lambda b$ is

$$\left|\frac{b \times (a_2 - a_1)}{|b|}\right|.$$

11.1.20 The vector equation of a plane which is at a distance p from the origin, where $\hat{n}$ is the unit vector normal to the plane, is $r.\hat{n}=p$.

11.1.21 Equation of a plane which is at a distance p from the origin with direction cosines of the normal to the plane as l, m, n is $lx + my + nz = p$.

11.1.22 The equation of a plane through a point whose position vector is a and perpendicular to the vector n is $(r - a).n=0$ or $r.n=d$, where $d=a.n$.

11.1.23 Equation of a plane perpendicular to a given line with direction ratios a, b, c and passing through a given point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

11.1.24 Equation of a plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

11.1.25 Vector equation of a plane that contains three non-collinear points having position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is $(\vec{r}-\vec{a}).\left[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right]=0$

11.1.26 Equation of a plane that cuts the co-ordinates axes at (a, 0, 0), (0, b, 0) and (0, 0, c) is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

11.1.27 Vector equation of any plane that passes through the intersection of planes $\vec{r}.\vec{n}_1=d_1$ and $\vec{r}.\vec{n}_2=d_2$ is $(\vec{r}.\vec{n}_1-d_1)+\lambda(\vec{r}.\vec{n}_2-d_2)=0$, where λ is any non-zero constant.

11.1.28 Cartesian equation of any plane that passes through the intersection of two given planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is $(A_1x + B_1y + C_1z + D_1) + \lambda\,(A_2x + B_2y + C_2z + D_2) = 0$.

11.1.29 Two lines $\vec{r} = \vec{a}_1+\lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2+\lambda\vec{b}_2$ are coplanar if $(\vec{a}_2-\vec{a}_1)\,.\,(\vec{b}_1\times\vec{b}_2)=0$

11.1.30 Two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0,$$

11.1.31 In vector form, if θ is the acute angle between the two planes, $\vec{r}\,.\,\vec{n}_1 = d_1$ and $\vec{r}\,.\,\vec{n}_2 = d_2$, then $\theta=\cos^{-1}\frac{|\vec{n}_1\,.\,\vec{n}_2|}{|\vec{n}_1|.|\vec{n}_2|}$

11.1.32 The acute angle θ between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and plane $\vec{r}\,.\,\vec{n} = d$ is given by

$$\sin\theta = \frac{|b \,.\, n|}{|b|\,.\,|n|}.$$

11.2 Solved Examples

Short Answer (S.A.)

Example 1 If the direction ratios of a line are 1, 1, 2, find the direction cosines of the line.

Solution The direction cosines are given by

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}},\ m = \frac{b}{\sqrt{a^2+b^2+c^2}},\ n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Here a, b, c are 1, 1, 2, respectively.

Therefore, $l = \frac{1}{\sqrt{1^2+1^2+2^2}},\ m = \frac{1}{\sqrt{1^2+1^2+2^2}},\ n = \frac{2}{\sqrt{1^2+1^2+2^2}}$

i.e., $l = \frac{1}{\sqrt{6}}, m = \frac{1}{\sqrt{6}}, n = \frac{2}{\sqrt{6}}$ i.e. $\pm\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ are D.C's of the line.

Example 2 Find the direction cosines of the line passing through the points P (2, 3, 5) and Q (–1, 2, 4).

Solution The direction cosines of a line passing through the points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are

$$\frac{x_2 - x_1}{\text{PQ}},\ \frac{y_2 - y_1}{\text{PQ}},\ \frac{z_2 - z_1}{\text{PQ}}.$$

Here $\text{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$$= \sqrt{(-1-2)^2 + (2-3)^2 + (4-5)^2} = \sqrt{9+1+1} = \sqrt{11}$$

Hence D.C.'s are