12.2 Coordinate Axes and Coordinate Planes in Three Dimensional Space

Consider three planes intersecting at a point O such that these three planes are mutually perpendicular to each other (Fig 12.1). These three planes intersect along the lines X′OX, Y′OY and Z′OZ, called the *x*, *y* and *z-axes*, respectively. We may note that these lines are mutually perpendicular to each other. These lines constitute the *rectangular coordinate system*. The planes XOY, YOZ and ZOX, called, respectively the XY-*plane*, YZ-*plane* and the ZX-*plane*, are known as the three coordinate planes. We take the XOY plane as the plane of the paper and the line Z′OZ as perpendicular to the plane XOY. If the plane of the paper is considered as horizontal, then the line Z′OZ will be vertical. The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ′ are taken as negative. Similarly, the distance measured to the right of ZX-plane along OY are taken as positive, to the left of ZX-plane and along OY′ as negative, in front of the YZ-plane along OX as positive and to the back of it along OX′ as negative. The point O is called the *origin* of the coordinate system. The three coordinate planes divide the space into eight parts known as *octants*. These octants could be named as XOYZ, X′OYZ, X′OY′Z, XOY′Z, XOYZ′, X′OYZ′, X′OY′Z′ and XOY′Z′. and denoted by I, II, III, ..., VIII , respectively.

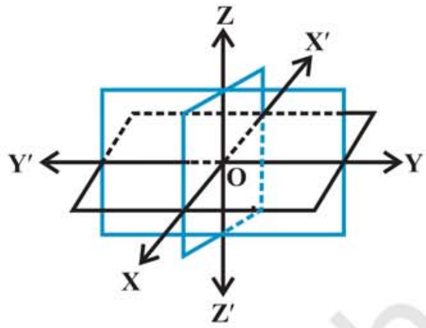

Fig 12.1

12.3 Coordinates of a Point in Space

Having chosen a fixed coordinate system in the space, consisting of coordinate axes, coordinate planes and the origin, we now explain, as to how, given a point in the space, we associate with it three coordinates (*x,y,z*) and conversely, given a triplet of three numbers (*x, y, z*), how, we locate a point in the space.

Given a point P in space, we drop a perpendicular PM on the XY-plane with M as the foot of this perpendicular (Fig 12.2). Then, from the point M, we draw a perpendicular ML to the *x*-axis, meeting it at L. Let OL be *x*, LM be *y* and MP be *z*. Then *x,y* and *z* are called the *x*, *y* and *z coordinates*, respectively, of the point P in the space. In Fig 12.2, we may note that the point P (*x*, *y*, *z*) lies in the octant XOYZ and so all *x*, *y*, *z* are positive. If P was in any other octant, the signs of *x*, *y* and *z* would change

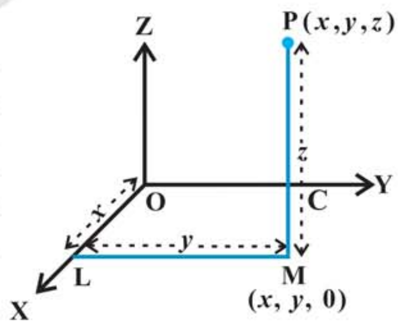

Fig 12.2

accordingly. Thus, to each point P in the space there corresponds an ordered triplet (x, y, z) of real numbers.

Conversely, given any triplet (x, y, z), we would first fix the point L on the x-axis corresponding to x, then locate the point M in the XY-plane such that (x, y) are the coordinates of the point M in the XY-plane. Note that LM is perpendicular to the x-axis or is parallel to the y-axis. Having reached the point M, we draw a perpendicular MP to the XY-plane and locate on it the point P corresponding to z. The point P so obtained has then the coordinates (x, y, z). Thus, there is a one to one correspondence between the points in space and ordered triplet (x, y, z) of real numbers.

Alternatively, through the point P in the space, we draw three planes parallel to the coordinate planes, meeting the x-axis, y-axis and z-axis in the points A, B and C, respectively (Fig 12.3). Let OA = x, OB = y and OC = z. Then, the point P will have the coordinates x, y and z and we write P (x, y, z). Conversely, given x, y and z, we locate the three points A, B and C on the three coordinate axes. Through the points A, B and C we draw planes parallel to the YZ-plane, ZX-plane and XY-plane, respectively. The point of interesection of these three planes, namely, ADPF, BDPE and CEPF is obviously the point P, corresponding to the ordered triplet (x, y, z). We observe that if P (x, y, z) is any point in the space, then x, y and z are perpendicular distances from YZ, ZX and XY planes, respectively.

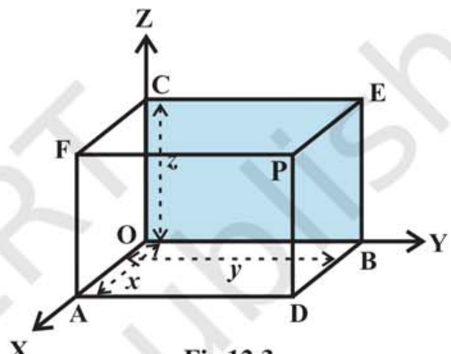

Fig 12.3

Note The coordinates of the origin O are (0,0,0). The coordinates of any point on the x-axis will be as (x,0,0) and the coordinates of any point in the YZ-plane will be as (0, y, z).

Remark The sign of the coordinates of a point determine the octant in which the point lies. The following table shows the signs of the coordinates in eight octants.

Table 12.1

Octants / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

$$CA^2 = (3 - 25)^2 + (6 + 41)^2 + (9 - 5)^2$$
$$= 484 + 2209 + 16 = 2709$$

We find that $CA^2 + AB^2 \neq BC^2$.

Hence, the triangle ABC is not a right angled triangle.

Example 6 Find the equation of set of points P such that $PA^2 + PB^2 = 2k^2$, where A and B are the points (3, 4, 5) and (–1, 3, –7), respectively.

Solution Let the coordinates of point P be (x, y, z).

Here $PA^2 = (x - 3)^2 + (y - 4)^2 + (z - 5)^2$

$PB^2 = (x + 1)^2 + (y - 3)^2 + (z + 7)^2$

By the given condition $PA^2 + PB^2 = 2k^2$, we have

$$(x - 3)^2 + (y - 4)^2 + (z - 5)^2 + (x + 1)^2 + (y - 3)^2 + (z + 7)^2 = 2k^2$$

i.e., $2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z = 2k^2 - 109$.

EXERCISE 12.2

1. Find the distance between the following pairs of points:
   (i) (2, 3, 5) and (4, 3, 1) (ii) (–3, 7, 2) and (2, 4, –1)
   (iii) (–1, 3, – 4) and (1, –3, 4) (iv) (2, –1, 3) and (–2, 1, 3).
2. Show that the points (–2, 3, 5), (1, 2, 3) and (7, 0, –1) are collinear.
3. Verify the following:
   (i) (0, 7, –10), (1, 6, – 6) and (4, 9, – 6) are the vertices of an isosceles triangle.
   (ii) (0, 7, 10), (–1, 6, 6) and (– 4, 9, 6) are the vertices of a right angled triangle.
   (iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8) and (2, –3, 4) are the vertices of a parallelogram.
4. Find the equation of the set of points which are equidistant from the points (1, 2, 3) and (3, 2, –1).
5. Find the equation of the set of points P, the sum of whose distances from A (4, 0, 0) and B (– 4, 0, 0) is equal to 10.

12.5 Section Formula

In two dimensional geometry, we have learnt how to find the coordinates of a point dividing a line segment in a given ratio internally. Now, we extend this to three dimensional geometry as follows:

Let the two given points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Let the point $R(x, y, z)$ divide PQ in the given ratio $m : n$ internally. Draw PL, QM and RN perpendicular to the XY-plane. Obviously PL || RN || QM and feet of these perpendiculars lie in a XY-plane. The points L, M and N will lie on a line which is the intersection of the plane containing PL, RN and QM with the XY-plane. Through the point R draw a line ST parallel to the line LM. Line ST will intersect the line LP externally at the point S and the line MQ at T, as shown in Fig 12.5.

Also note that quadrilaterals LNRS and NMTR are parallelograms.

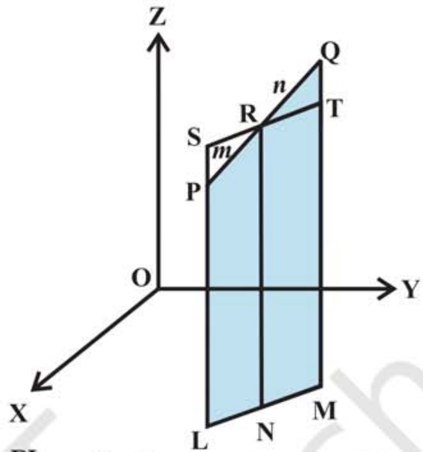

Fig 12.5

The triangles PSR and QTR are similar. Therefore,

$$\frac{m}{n} = \frac{\text{PR}}{\text{QR}} = \frac{\text{SP}}{\text{QT}} = \frac{\text{SL} - \text{PL}}{\text{QM} - \text{TM}} = \frac{\text{NR} - \text{PL}}{\text{QM} - \text{NR}} = \frac{z - z_1}{z_2 - z}$$

This implies $$z = \frac{mz_2 + nz_1}{m + n}$$

Similarly, by drawing perpendiculars to the XZ and YZ-planes, we get

$$y = \frac{my_2 + ny_1}{m + n} \text{ and } x = \frac{mx_2 + nx_1}{m + n}$$

Hence, the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

If the point R divides PQ externally in the ratio $m : n$, then its coordinates are obtained by replacing n by $-n$ so that coordinates of point R will be

$$\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n}\right)$$

Case 1 Coordinates of the mid-point: In case R is the mid-point of PQ, then

$m : n = 1 : 1$ so that $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$ and $z = \frac{z_1 + z_2}{2}$.

These are the coordinates of the mid point of the segment joining P (x_1, y_1, z_1) and Q (x_2, y_2, z_2).

Case 2 The coordinates of the point R which divides PQ in the ratio $k : 1$ are obtained by taking $k = \frac{m}{n}$ which are as given below:

$$\left(\frac{kx_2 + x_1}{1+k}, \frac{ky_2 + y_1}{1+k}, \frac{kz_2 + z_1}{1+k}\right)$$

Generally, this result is used in solving problems involving a general point on the line passing through two given points.

Example 7 Find the coordinates of the point which divides the line segment joining the points (1, –2, 3) and (3, 4, –5) in the ratio 2 : 3 (i) internally, and (ii) externally.

Solution (i) Let P (x, y, z) be the point which divides line segment joining A(1, – 2, 3) and B (3, 4, –5) internally in the ratio 2 : 3. Therefore

$$x = \frac{2(3) + 3(1)}{2+3} = \frac{9}{5},\ y = \frac{2(4) + 3(-2)}{2+3} = \frac{2}{5},\ z = \frac{2(-5) + 3(3)}{2+3} = \frac{-1}{5}$$

Thus, the required point is $\left(\frac{9}{5}, \frac{2}{5}, \frac{-1}{5}\right)$

(ii) Let P (x, y, z) be the point which divides segment joining A (1, –2, 3) and B (3, 4, –5) externally in the ratio 2 : 3. Then

$$x = \frac{2(3) + (-3)(1)}{2 + (-3)} = -3,\quad y = \frac{2(4) + (-3)(-2)}{2 + (-3)} = -14,\ z = \frac{2(-5) + (-3)(3)}{2 + (-3)} = 19$$

Therefore, the required point is (–3, –14, 19).

Example 8 Using section formula, prove that the three points (– 4, 6, 10), (2, 4, 6) and (14, 0, –2) are collinear.

Solution Let A (– 4, 6, 10), B (2, 4, 6) and C(14, 0, – 2) be the given points. Let the point P divides AB in the ratio $k : 1$. Then coordinates of the point P are

$$\left(\frac{2k-4}{k+1}, \frac{4k+6}{k+1}, \frac{6k+10}{k+1}\right)$$

Let us examine whether for some value of k, the point P coincides with point C.

On putting $\frac{2k-4}{k+1} = 14$, we get $k = -\frac{3}{2}$